Reductions in computability theory from a constructive point of view

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1. Thanks to organizers for the invitation.
2. The first part of the work is joint with Kazuto Yoshimura from JAIST.

1. Instance reducibility
2. Other reducibilities

1. The talk consists of two parts
2. I will first talk about instance reducibility, a natural notion of reducibility in constructive mathematics (it trivializes to implication classically), and its connection to Weihrauch reducibility, which has been studied in some detail recently by various people (is Arno in the audience?).
3. Then I will discuss some work in progress: how to deal with other reducibilities: many-to-one, truth-table, and Turing reducibilities.

Synthetic mathematics:

- build a model to taste,
- argue on "high level" internally,
- hide nitty-gritty details in the model

1. I am not after just *any* way of constructivizing these topics. In order for it to be worth it, the constructivization must result in what I would call *natural* mathematics.
2. For instance, I do not wish to speak in detail about Turing machines in the constructive setting – these should be hidden inside a model, such as Kleene's realizability.
3. Rather, the concepts and the theorems should expose a conceptual, or high-level ideas, or relate known results in computability theory to standard notions and theorems in analysis and topology.
4. This is called *synthetic* because we synthesize a model in such a way that its internal language, that is the mathematics inside the model, does what we want elegantly (we hope!), while hiding nitty-gritty details under the hood.
5. But you will see what I mean when I do it. Well known examples of this approach are non-standard analysis and synthetic differential geometry.

$$(\forall y \in B \,.\, \psi(y)) \Rightarrow \forall x \in A \,.\, \phi(x)$$

1. In constructive mathematics, and generally in all mathematics, we often want to prove that one universal statement implies another.
2. Note, there is no restriction on ϕ and ψ here.
3. What's a common way of proving such statements? To answer this, let's look at an example. And let's make it an exercise in constructive reasoning.

Show that 1. implies 2.:

1. $\forall x \in \mathbb{R} \,.\, x = 0 \lor \neg(x = 0)$
2. $\forall f \in \{0,1\}^{\mathbb{N}} \,.\, (\forall n \,.\, f(n) = 0) \lor \neg(\forall n \,.\, f(n) = 0)$

1. Let us show that statement 1 implies statement 2.
2. Statement 1 says that every real is zero or not zero.
3. Statement 2 says that every infinite binary sequence is all zeroes or not.
4. If you think about this for yourself, or if you have seen it in a book, the proof looked somewhat as follows.
5. Let us note the form of the proof: given an instance f of the second statement we find a *suitable* instance x of the first statement, such that the first statement at x implies the second statement at f.
6. Let us give the technique a name.

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Solution: given $f : \mathbb{N} \to \{0,1\}$ define

$$x = \sum_{n=0}^{\infty} f(n) \cdot 2^{-n}.$$

Either $x = 0$ or $x \neq 0$. In the first case it follows that $\forall n \,.\, f(n) = 0$, and in the second $\neg\forall n \,.\, f(n) = 0$.

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6. Let us give the technique a name.

Definition

A predicate $\phi \subseteq A$ is *instance reducible* to $\psi \subseteq B$, written $\phi \leq_{\mathrm{I}} \psi$, if there is a total relation $K \subseteq A \times B$ such that

$$\forall x \in A \,.\, (\exists y \in B \,.\, K(x, y) \land \psi(y)) \Rightarrow \phi(x). \qquad (1)$$

Say that y *suitable for* x when $K(x, y)$.

1. I will equate predicates with subsets, or subobjects, i.e., they are *not* formulas (only a logician would think that).
2. The definition reflects the solution on previous slide, where "suitable" means is captured by the relation K.
3. Actually, on the previous slide K was a function because we found a specific suitable y for a given x. This is often the case, but in general K need not be single valued.
4. Observe that we can rewrite the defining condition as a negative formula (not containing $\exists$). This says that the computational content of an instance reducibility is "stored" only in K.

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Note: condition (1) is equivalent to

$$\forall x \in A \,.\, \forall y \in B \,.\, K(x,y) \wedge \psi(y) \Rightarrow \phi(x).$$

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Theorem

If $\phi \leq_{\mathrm{I}} \psi$ then $(\forall y \in B \,.\, \psi(y)) \Rightarrow \forall x \in A \,.\, \phi(x)$.

Proof. Given $x \in A$, there is $y \in B$ such that $K(x, y)$. By assumption we also have $\psi(y)$ therefore $\phi(x)$. □

1. Instance reducibility is indeed sufficient to show the implication between the corresponding universally quantified statements.
2. I am not going through the proof, it's very simple.
3. Note the reversal of order, we have "ϕ is less than ψ" but "ψ implies ϕ". This is in accordance with the idea that a notion of reduction measures how difficult a problem is, not how easy.
4. We may ask whether the converse holds. It does classically, but not constructively. Under further conditions, studied by Kazuto, it is sometimes possible to obtain the converse. This then gives us separation results in constructive reverse math, ask me later if you're interested.

Theorem

Instance reducibilities form a distributive lattice.

1. The basic structure of instance reducibility is described by the following theorem. By lattice we mean a bounded one, i.e., it has bottom and top.
2. The lattice structure is straightforward and the properties easy to check.

Theorem

Instance reducibilities form a distributive lattice.

Proof. The operations are as follows:

- The bottom is $\emptyset \subseteq \emptyset$.
- The top is $\emptyset \subseteq \{\star\}$.
- The supremum of $\phi \subseteq A$ and $\psi \subseteq B$ is $\phi + \psi \subseteq A + B$ where for $x \in A$ and $y \in B$

$$(\phi + \psi)(x) \iff \phi(x) \quad \text{and} \quad (\phi + \psi)(y) \iff \psi(y)$$

- The infimum of $\phi \subseteq A$ and $\psi \subseteq B$ is $\phi \times \psi \subseteq A \times B$ where

$$(\phi \times \psi)(x, y) \iff \phi(x) \vee \psi(y)$$

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Given $\phi \subseteq A$ and B define $\phi^B \subseteq A^B$ by

$$\phi^B(f) \iff \forall y \in B \,.\, \phi(f(y)).$$

Then $\phi \leq_{\mathrm{I}} \psi^B$ means that ϕ reduces to B-many instances of ψ.

1. Let us look at a couple of other constructions on instance reducibilities.
2. The first one is parameterization. It allows us to reduce to many instances rather than just one.
3. For example, we can set B to $\mathbb{N}$ to get "countably many instances".
4. A slightly more complicated construction in the style of Kleene iteration gives "finitely many instances".

Given $f : A \to B$ and $\psi \subseteq B$, define $f^*\psi \subseteq A$ by

$$f^*\psi(x) \iff \psi(f(x)).$$

1. Given a function $f : A \to B$ we can pull back a predicate from B to A. This is just the preimage of ψ under f.
2. In the other direction we have two options: one uses a universal quantifier and the other the existential one.
3. They correspond to the preimage satisfying the original predicate universally or existentially.
4. We have a basic inequalities, where two of them hold provided that f is onto.
5. The useful case is when f is a projection from $A \times B$ to A with inhabited B. The formulas then correspond to usual quantifications.

Given $f : A \to B$ and $\psi \subseteq B$, define $f^*\psi \subseteq A$ by

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Given $\phi \subseteq A$, define $\forall_f \phi \subseteq B$ and $\exists_f \phi \subseteq B$ by

$$\forall_f \phi(y) \iff \forall x \in A \,.\, f(x) = y \Rightarrow \phi(x)$$
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Then:

$$f^*\psi \leq_{\mathrm{I}} \psi \quad \text{and} \quad \phi \leq_{\mathrm{I}} \forall_f \phi.$$

If f is surjective then also

$$\psi \leq_{\mathrm{I}} f^*\psi \quad \text{and} \quad \exists_f \phi \leq_{\mathrm{I}} \phi.$$

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Example: $f = \pi_1 : A \times B \to A$ with B inhabited.

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Theorem

There is an antimonotone embedding of truth values into instance reducibilities.

Proof. A truth value p corresponds to the predicate $\phi_p \subseteq \{\star\}$ defined by $\phi_p(x) \iff p$.
We have $p \Rightarrow q$ iff $\phi_q \leq_{\mathrm{I}} \phi_p$. □

1. Instance reducibilities form a very rich structure that contains many others.
2. The truth values embed into instance reducibilities.
3. We shall see in a moment that instance reducibility corresponds to Weihrauch reducibility in a certain realizability model. The embedding of Prop there becomes the embedding of the Medvedev lattice into Weihrauch lattice.

Theorem

Define $\top_A$ *to be the top predicate* $A \subseteq A$.

1. $\phi \leq_{\mathrm{I}} \top_{\{\star\}}$ *iff* $\phi = \top_A$ *for some* A,
2. $\top_A \leq_{\mathrm{I}} \top_B$ *iff there is a total* $K \subseteq A \times B$.

1. Another large part of the instance reducibility are sets under "total relation" ordering.
2. The condition that there is a total $K \subseteq A \times R$ can be read as "A is at least as large as B".

For $F \subseteq \mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}}$ define

$$F[\alpha] = \{\beta \in \mathbb{N}^{\mathbb{N}} \mid (\alpha, \beta) \in F\},$$
$$\|F\| = \{\alpha \in \mathbb{N}^{\mathbb{N}} \mid \exists \beta \,.\, (\alpha, \beta) \in F\}.$$

Definition

A subset $F \subseteq \mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}}$ is *Weihrauch reducible* to $G \subseteq \mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}}$, written $F \leq_W G$, if there exist partial computable maps $k, \ell : \mathbb{N}^{\mathbb{N}} \hookrightarrow \mathbb{N}^{\mathbb{N}}$ such that, for all $\alpha, \beta \in \mathbb{N}^{\mathbb{N}}$:

1. if $\alpha \in \|F\|$ then $k(\alpha)$ is defined and $k(\alpha) \in \|G\|$,
2. if $\alpha \in \|F\|$ and $\beta \in G[k(\alpha)]$ then $\ell(\langle \alpha, \beta \rangle)$ is defined and $\ell(\langle \alpha, \beta \rangle) \in F[\alpha]$.

1. Let us now relate instance reducibility to a known notion of reducibility in computability theory, namely Weihrauch reducibility, which we recall.
2. Think of F as a description of a problem, α as a question, and β as an answer. Read $(\alpha, \beta) \in F$ as "β is an answer to question α for problem description F."
3. Under this reading $F[\alpha]$ is the set of F-answers to question α, and $\|F\|$ is the set of those questions that have an answer.
4. Many natural problems in computable analysis can be expressed in this form, because questions and answers are infinite sequences that can encode reals and complex numbers, continuous and smooth maps, open and closed sets, etc.
5. In the definition of $F \leq_W G$, the map k translates an F-question to a G-question, and ℓ translates a G-answer to an F-answer (it is also provided the original question).

For $F \subseteq \mathbb{N}^\mathbb{N} \times \mathbb{N}^\mathbb{N}$ define

$$F[\alpha] = \{\beta \in \mathbb{N}^\mathbb{N} \mid (\alpha, \beta) \in F\},$$
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Definition

An *extended Weihrauch degree* is a pair (U, F) where $U \subseteq \mathbb{N}^\mathbb{N}$ and $F \subseteq U \times \mathbb{N}^\mathbb{N}$. An (extended) *Weihrauch reduction* $(U, F) \leq_W (V, G)$ is given by partial computable maps $k, \ell : \mathbb{N}^\mathbb{N} \hookrightarrow \mathbb{N}^\mathbb{N}$ such that:

1. if $\alpha \in U$ then $k(\alpha)$ is defined and $k(\alpha) \in V$,
2. if $\alpha \in U$ and $\beta \in G[k(\alpha)]$ then $\ell(\alpha, \beta)$ is defined and $\ell(\langle \alpha, \beta \rangle) \in F[\alpha]$.

1. When we interpret instance reducibility in the realizability model which corresponds to TTE (relative function realizability), we do *not* get Weihrauch reducibility, but an extension of it.
2. The difference is as follows. Weihrauch reducibility only cares about questions that have an answer. In the extended version, we specify which questions are "valid", even though they may not have an answer. The reducibility must then translate *all* valid questions, not just those that have an answer.
3. We can actually explain how the definition arises: U is the set of realizers of elements of A, F is encodes ϕ, k is the realizer for totality of K, and ℓ is the realizer for implication $\psi(y) \wedge K(x, y) \Rightarrow \phi(x)$.
4. There are natural examples of extended degrees which are not proper degrees, for instance formal Church's thesis CT: $(\alpha, \beta) \in CT$ iff $\beta(0)$ is the code of a machine computing α.

Theorem

Instance reducibility corresponds to extended Weihrauch reducibility.

Precisely: the lattice of instance reducibilities interpreted in relative realizability topos $\mathsf{RT}((\mathbb{N}^{\mathbb{N}})_{\mathsf{eff}}, \mathbb{N}^{\mathbb{N}})$ is equivalent to the extended Weihrauch lattice.

1. We can now state the exact correspondence between instance reducibility and extended Weihrauch reducibility. It is simply this: when instance reducibility is interpreted in the realizability topos of Baire-space representations and computable maps we obtain the extended Weihrauch lattice. This topos is the setup for TTE and also the standard model for Brouwerian intuitionism.
2. And if we restrict to $\neg\neg$-dense predicates, we obtain the Weihrauch lattice.
3. This is how synthetic mathematics is supposed to work. The technicalities are hidden in the model, so we are left with a clean concept and clean proofs.
4. The benefits are immediate: proofs in constructive reverse mathematics yield Weihrauch reducibilities. In the opposite direction, non-existence of Weihrauch reductions implies non-provability of instance reductions.
5. Furthermore, we can interpret the definition in other realizability models, and even in sheaf toposes, etc.

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Theorem

Instance reducibility restricted to $\neg\neg$-dense predicates corresponds to Weihrauch reducibility.

(A predicate $\phi \subseteq A$ is $\neg\neg$-*dense* when $\neg\exists x \in A \,.\, \neg\phi(x)$.)

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1. Instance reducibility
2. Other reducibilities

1. This brings us to the second part of the talk.
2. We would like to do something similar for other kinds of reducibility.

Definition

A predicate $\phi \subseteq A$ is *many-to-one reducible* to $\psi \subseteq B$, written $\phi \leq_{\mathrm{M}} \psi$, if there is $f : A \to B$ such that $\phi = f^{*}\psi$.

1. First of all, a many-to-one reduction is just an inverse image map.
2. We have again a simple and mathematically clean definition which mentions no computability. It works for arbitrary predicates on arbitrary sets, not just subsets of $\mathbb{N}$.
3. Next, truth-table reductions correspond to instance reducibility (from finitely many instances, rather than just one) interpreted in Kleene's number realizability.
4. But Turing reductions are not so easy.

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Theorem

Instance reducibility corresponds to truth-table reducibility.

Precisely: when instance reducibility (allowing reductions to finitely many instances) is interpreted in Kleene's number realizability, it restricts to truth-table reducibility on subsets of $\mathbb{N}$.

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3. Next, truth-table reductions correspond to instance reducibility (from finitely many instances, rather than just one) interpreted in Kleene's number realizability.
4. But Turing reductions are not so easy.

- A *partial oracle* is a pair (A_0, A_1) of disjoint subsets of $\mathbb{N}$. The space of all partial oracles:

$$\mathbb{T} = \{(A_0, A_1) \in \mathcal{P}(\mathbb{N}) \times \mathcal{P}(\mathbb{N}) \mid A_0 \cap A_1 = \emptyset\}.$$

- We order $\mathbb{T}$ by

$$(A_0, A_1) \leq (B_0, B_1) \iff A_0 \subseteq B_0 \land A_1 \subseteq B_1.$$

  This is *Plotkin's universal domain*, a dcpo whose finite elements are pairs of finite disjoint subsets.

- The *total oracles* are the maximal elements of $\mathbb{T}$. They are precisely those (A_0, A_1) for which $A_0 = \mathbb{N} \setminus A_1$ and $A_1 = \mathbb{N} \setminus A_0$.

1. We need to consider *partial* oracles, i.e., ones that do not necessarily give an answer: a pair of disjoint sets (A_0, A_1). Think of A_0 as the questions with answers "no" and A_1 with answer "yes". Since the union of A_0 and A_1 need not be all of $\mathbb{N}$ the oracle is partial.
2. There is an obvious order on partial oracles which turns $\mathbb{T}$ into a directed-complete poset. It is in fact a well-known object domain theory: Plotkin's universal domain.
3. Because every subsets of $\mathbb{N}$ is a directed union of its finite subsets, $\mathbb{T}$ is generated by pairs of finite disjoint sets.
4. It would be too restrictive to say that an oracle (A_0, A_1) is total when $A_0 \cup A_1 = \mathbb{N}$, that would give only computable oracle. We need to take as total the maximal elements of $\mathbb{T}$. They are precisely pairs (A_0, A_1) that are complements of each other.

Let $\mathcal{K}(\mathbb{T})$ be the set of finite elements of $\mathbb{T}$.

Definition

A *(partial) Turing reduction* is a continuous map $r : \mathbb{T} \to \mathbb{T}$ whose graph

$$\{(x, y) \in \mathcal{K}(\mathbb{T}) \times \mathcal{K}(\mathbb{T}) \mid y \leq r(x)\}$$

is countable.

1. What should we take as a reduction? Obviously, some sort of a map $\mathbb{T} \to \mathbb{T}$, but with what properties?
2. It should be continuous, i.e., preserve directed suprema. Then it will already be determined by its values on the finite oracles, which is our way of expressing the use principle from computability. We see the synthetic approach at work: we use principle becomes continuity – an analogy that becomes an exact correspondence.
3. Continuity is not enough. Every constant map is continuous, but we certainly do not want to be able to reduce all oracles to every oracle. We need to express the fact that a reduction is Turing computable.
4. Recall that a number-theoretic function is computable iff its graph is c.e. Internal the c.e. condition is just “countable”. So a reduction is a continuous map with a countable graph.
5. We are of course talking about *partial* reductions. A total one would map total oracles to total oracles.

- For a sequence of Turing reductions $(r_n : \mathbb{T} \to \mathbb{T})_{n \in \mathbb{N}}$, let the *Post-Turing statement* $\mathsf{PT}(r)$ be

$$\exists x, y \in \mathsf{Max}(\mathbb{T}) \,.\, \forall n \in \mathbb{N} \,.\, x \neq r_n(y) \land y \neq r_n(x)$$

  In words: "There are total oracles x and y which are not reduced to each other by the reductions $(r_n)_n$."
- Let PT be the statement $\forall (r_n)_n \,.\, \mathsf{PT}(r)$: "$\mathsf{PT}(r)$ holds for all sequences of reductions $(r_n)_n$."
- Limited principle of omniscience (LPO):

$$\forall f \in \{0,1\}^{\mathbb{N}} \,.\, (\exists n \,.\, f(n) = 1) \lor \neg(\exists n \,.\, f(n) = 1)$$

  An equivalent form: given $n \in \mathbb{N}$ and countable $B \subseteq \mathbb{N}$, either $n \in B$ or $n \notin B$.

1. Let us tackle the Post-Turing theorem which says that there are incomparable *total* oracles.
2. We shall prove the theorem with respect to any given sequence of reductions. When we specialize to all the computable reductions in the effective topos, we get the classical theorem.
3. Actually, we are not going to prove $\mathcal{PT}$, but rather reduce it to countably many instances of the Limited Principle of Omniscience.

Theorem (Post-Turing)

$\mathsf{PT} \leq_{\mathrm{I}} \mathrm{LPO}^{\mathbb{N}}$.

1. Caveat: we need to use a formulation of metric spaces in which distances are measured by *upper* reals, i.e., we only get upper bounds on distances.
2. And we need a carefully crafted version of the Baire category theorem, ask me for details.
3. It would be interesting to get a corresponding theorem in the Weihrauch lattice in the context of TTE, perhaps it is already known?
4. How about the Friedberg-Muchnik theorem? I do not know.

Theorem (Post-Turing)

$\mathsf{PT} \leq_{\mathrm{I}} \mathrm{LPO}^{\mathbb{N}}$.

Proof outline. Total oracles $\mathsf{Max}(\mathbb{T})$ form a complete metric space. We apply the Baire category theorem to the sets

$$U_n = \{(x, y) \in \mathsf{Max}(\mathbb{T}) \times \mathsf{Max}(\mathbb{T}) \mid r_n(x) \neq y\},$$
$$V_n = \{(x, y) \in \mathsf{Max}(\mathbb{T}) \times \mathsf{Max}(\mathbb{T}) \mid r_n(y) \neq x\}.$$

We need countably many instances of LPO to show that the U_n's and V_n's are actually open dense subsets. □

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