

On the Failure of Fixed-Point Theorems for Chain-complete Lattices in the Effective Topos

Andrej Bauer
Faculty of Mathematics and Physics
University of Ljubljana
Ljubljana, Slovenia
Email: Andrej.Bauer@andrej.com

January 25, 2009

Abstract

In the effective topos there exists a chain-complete distributive lattice with a monotone and progressive endomap which does not have a fixed point. Consequently, the Bourbaki-Witt theorem and Tarski's fixed-point theorem for chain-complete lattices do not have constructive (topos-valid) proofs.

1 Introduction

Fixed-point theorems state that maps have fixed points under certain conditions. They are used prominently in denotational semantics, for example to give meaning to recursive programs. In fact, it is hard to overestimate their applicability and importance in mathematics in general.

A constructive proof of a fixed-point theorem makes the theorem twice as worthy because it yields an algorithm for computing a fixed point. Indeed, many fixed-point theorems have constructive proofs, of which we might mention Lawvere's fixed-point theorem [5], Tarski's fixed-point theorem for a monotone map on a complete lattice [7], and Pataraia's generalization of it to directed-complete posets [6]. Two that have defied constructive proofs are Tarski's theorem for chain-complete posets and the Bourbaki-Witt theorem [1, 9] for progressive maps on chain-complete posets, see Section 5 for their precise statements.

I show that in the effective topos [3] there is a chain-complete distributive lattice with a monotone and progressive endomap which does *not* have a fixed point. An immediate consequence of this is that both Tarski's theorem for chain-complete posets and the Bourbaki-Witt theorem have no constructive (topos-valid) proofs.

The outline of the argument is as follows. In the effective topos \mathbf{Eff} every chain is a quotient of a subobject of the natural numbers, hence it has at most countably many global points. Consequently, the poset ω_1 of set-theoretic countable ordinals is chain-complete in the effective topos, even though it is only countably complete in the topos of sets. The successor function on ω_1 is monotone, progressive, and does not have a fixed point. We work out the details of the preceding argument carefully in order not to confuse external and internal notions of chain-completeness and countability. We use [8] as a reference on the effective topos. For the uninitiated, we have included a brief overview of the effective topos in Appendix A.

2 Discrete objects in the effective topos

An object in the effective topos is *discrete*¹ when it is a quotient of a subobject of the natural numbers object \mathbb{N} . Such objects were studied in [4], where it is shown that X is discrete precisely when it is orthogonal to 2 , by which we mean that the diagonal map $X \rightarrow X^{\nabla 2}$ is constant. Here $2 = \{0, 1\}$ is the two-element set and $\nabla : \mathbf{Set} \rightarrow \mathbf{Eff}$ is the “constant objects” functor, see Appendix A.3. In the internal language of \mathbf{Eff} discreteness of X is expressed by the statement

$$\forall f \in X^{\nabla 2} . \forall p \in \nabla 2 . f(p) = f(1), \quad (1)$$

which says that every $f : \nabla 2 \rightarrow X$ is constant. We are interested in the object $\mathcal{D}(X)$ of discrete subobjects of X , which we define in the internal language as

$$\mathcal{D}(X) = \{A \in \mathcal{P}(X) \mid A \perp \nabla 2\},$$

where $\mathcal{P}(X)$ is the powerobject and $A \perp \nabla 2$ is the statement²

$$\forall f \in X^{\nabla 2} . (\forall p \in \nabla 2 . f(p) \in A) \implies (\forall p \in \nabla 2 . f(p) = f(1)). \quad (2)$$

Let us explicitly compute $\mathcal{D}(X)$ in case $X = \nabla S$ for a set S . The powerobject $\mathcal{P}(\nabla S)$ is the set $P(\mathbb{N})^S$ with the non-standard equality predicate

$$[A =_{\mathcal{P}(\nabla S)} B] = (A \Rightarrow B) \wedge (B \Rightarrow A).$$

The object $\mathcal{D}(\nabla S)$ is the set $P(\mathbb{N})^S$ with non-standard equality predicate

$$[A =_{\mathcal{D}(\nabla S)} B] = (A \Rightarrow B) \wedge (B \Rightarrow A) \wedge D(A),$$

¹The terminology is established and somewhat unfortunate, as it falsely suggests that a discrete object has decidable equality.

²We take care not to assume that a variable A ranging over a powerobject $\mathcal{P}(X)$ is an actual object in the topos, which is why (1) and (2) differ slightly.

where $D : P(\mathbb{N})^S \rightarrow P(\mathbb{N})$ is a strict extensional relation representing the predicate (2). To compute D we recall how universal quantification over a constant object works.

Suppose T is a set, X is an object, and ϕ is a formula with free variables t and x ranging over ∇T and X , respectively, represented by the strict extensional relation $F : T \times |X| \rightarrow P(\mathbb{N})$. Then the predicate $\forall t \in \nabla T. \phi$ is represented by the strict extensional relation $|X| \rightarrow P(\mathbb{N})$ defined by

$$x \mapsto \bigcap_{t \in T} F(t, x).$$

When we apply this to the universal quantifiers in (2), and use the fact that $\nabla S^{\nabla 2}$ is isomorphic to $\nabla(S^2)$, we find after a short calculation that

$$\begin{aligned} D(A) &= \bigcap_{f \in S^2} A(f(0)) \cap A(f(1)) \Rightarrow [f(0) =_{\nabla S} f(1)] \\ &= \bigcap_{(x,y) \in S^2} A(x) \cap A(y) \Rightarrow [x =_{\nabla S} y]. \end{aligned}$$

We will need to know precisely when $D(A)$ is non-empty. If $x \neq y$ then $A(x) \cap A(y) \Rightarrow [x =_{\nabla S} y]$ is inhabited only if $A(x) \cap A(y) = \emptyset$, because $x \neq y$ implies $[x =_{\nabla S} y] = \emptyset$. Thus a necessary condition for $D(A)$ to be non-empty is that $x \neq y$ implies $A(x) \cap A(y) = \emptyset$. But this condition is also sufficient, since it implies that

$$\begin{aligned} D(A) &= \bigcap_{(x,y) \in S^2} A(x) \cap A(y) \Rightarrow [x =_{\nabla S} y] = \\ &= \left(\bigcap_{x=y} A(x) \cap A(y) \Rightarrow [x =_{\nabla S} y] \right) \cap \left(\bigcap_{x \neq y} A(x) \cap A(y) \Rightarrow [x =_{\nabla S} y] \right) = \\ &= \left(\bigcap_{x=y} A(x) \Rightarrow \mathbb{N} \right) \cap \left(\bigcap_{x \neq y} \emptyset \Rightarrow \emptyset \right) = \left(\bigcap_{x=y} A(x) \Rightarrow \mathbb{N} \right) \end{aligned}$$

is non-empty because it contains at least (the Gödel codes of) the constant function $n \mapsto 0$.

Let $\text{cl}_{\neg} : \mathcal{P}(\nabla S) \rightarrow \nabla P(S)$ be the operator which composes with double negation $\neg\neg : \Omega \rightarrow \nabla 2$:

$$\mathcal{P}(\nabla S) \xrightarrow{\cong} \Omega^{\nabla S} \xrightarrow{\neg\neg^{\nabla S}} (\nabla 2)^{\nabla S} \xrightarrow{\cong} \nabla P(S)$$

Let $P_{\omega}(S)$ be the set of all countable subsets of a set S .

Proposition 2.1 *For any set S , the restriction of cl_{\neg} to $\mathcal{D}(\nabla S)$ factors through $\nabla P_{\omega}(S)$:*

$$\begin{array}{ccc} \mathcal{D}(\nabla S) & \xrightarrow{i} & \mathcal{P}(\nabla S) \\ \downarrow & & \downarrow \text{cl}_{\neg} \\ \nabla P_{\omega}(S) & \xrightarrow{\nabla j} & \nabla P(S) \end{array}$$

Proof. In the diagram above i and j are inclusions $\mathcal{D}(\nabla S) \subseteq \mathcal{P}(\nabla S)$ and $P_\omega(S) \subseteq P(S)$, respectively. Because ∇ is right adjoint to the global points functor Γ , and $\Gamma \circ \nabla$ is naturally isomorphic to the identity, there is a unique $c : \Gamma(\mathcal{D}(\nabla S)) \rightarrow P(S)$ such that $\text{cl}_{\neg\neg} \circ i$ is the composition of ∇c and the unit of the adjunction η :

$$\begin{array}{ccc} \mathcal{D}(\nabla S) & \xrightarrow{\eta} & \nabla\Gamma(\mathcal{D}(S)) \\ & \searrow \text{cl}_{\neg\neg} \circ i & \downarrow \nabla c \\ & & \nabla P(S) \end{array}$$

It suffices to show that c factors through j , since then $\text{cl}_{\neg\neg} \circ i = \nabla c \circ \eta$ factors through ∇j .

A global point $[A] : 1 \rightarrow \mathcal{D}(\nabla S)$ is represented by $A : S \rightarrow P(\mathbb{N})$ such that $D(A) \neq \emptyset$. Because $\text{cl}_{\neg\neg}$ is composition with $\neg\neg$, we get

$$c([A]) = \{x \in S \mid A(x) \neq \emptyset\}.$$

Earlier we established that $D(A) \neq \emptyset$ implies $A(x) \cap A(y) = \emptyset$ whenever $x \neq y$. Therefore, for each $n \in \mathbb{N}$ there is at most one $x \in A$ such that $n \in A(x)$, which means that there are at most countably many $x \in S$ for which $A(x) \neq \emptyset$. But then $c([A])$ is a countable subset of S , which is what we wanted to prove. \square

We shall need one more piece of knowledge about discrete objects. Define the object $B = (\{0, 1\}, =_B)$ to have the equality predicate

$$[x =_B y] = \begin{cases} \{0\} & \text{if } x = y = 0, \\ \{1\} & \text{if } x = y = 1, \\ \emptyset & \text{otherwise.} \end{cases}$$

The object B is isomorphic to $1 + 1$. By the *uniformity principle* [8, 3.2.21], the following statement is valid in the internal language of **Eff**: for all $\phi \in \mathcal{P}(\nabla 2 \times B)$, if $\forall p \in \nabla 2. \exists d \in B. \phi(p, d)$ then $\exists d \in B. \forall p \in \nabla 2. \phi(p, d)$. We require the following equivalent form.

Lemma 2.2 *The following statement is valid in the internal language of Eff: for all $\phi, \psi : \nabla 2 \rightarrow \Omega$, if $\forall p \in \nabla 2. (\phi(p) \vee \psi(p))$ then $\forall p \in \nabla 2. \phi(p)$ or $\forall p \in \nabla 2. \psi(p)$.*

Proof. We argue in the internal language of **Eff**. If $\forall p \in \nabla 2. (\phi(p) \vee \psi(p))$ then

$$\forall p \in \nabla 2. \exists d \in 2. ((d = 0 \wedge \phi(p)) \vee (d = 1 \wedge \psi(p))).$$

To see this, take $d = 0$ if $\phi(p)$ holds and $d = 1$ if $\psi(p)$ holds. By the uniformity principle

$$\exists d \in 2. \forall p \in \nabla 2. ((d = 0 \wedge \phi(p)) \vee (d = 1 \wedge \psi(p))).$$

Consider such $d \in 2$. If $d = 0$ then $\forall p \in \nabla 2. \phi(p)$, and if $d = 1$ we obtain $\forall p \in \nabla 2. \psi(p)$. \square

3 Posets and Chains in the Effective Topos

In this section we work entirely in the internal language of the effective topos. First we recall several standard order-theoretic notions. A *poset* (L, \leq) is an object L with a relation \leq which is reflexive, transitive, and antisymmetric. A *lattice* (L, \leq, \wedge, \vee) is a poset in which every elements $x, y \in L$ have a greatest lower bound $x \wedge y$, and least upper bound $x \vee y$. Note that a lattice need not have the smallest and the greatest element. A lattice is *distributive* if \wedge and \vee satisfy the distributivity laws $(x \wedge y) \vee z = (x \vee z) \wedge (y \vee z)$ and $(x \vee y) \wedge z = (x \wedge z) \vee (y \wedge z)$. An endomap $f : L \rightarrow L$ on a poset (L, \leq) is *monotone* when

$$\forall x, y \in L. (x \leq y \implies f(x) \leq f(y)),$$

and *progressive* when $\forall x \in L. x \leq f(x)$.

For $x \in L$ and $A \in \mathcal{P}(L)$ define $\text{bound}(x, A)$ to be the relation

$$\text{bound}(x, A) \iff \forall y \in L. (y \in A \implies y \leq x).$$

We say that $z \in L$ is the *supremum* of $A \in \mathcal{P}(L)$ when

$$\text{bound}(z, A) \wedge \forall y \in L. (\text{bound}(y, A) \implies y \leq z).$$

Lemma 3.1 *Suppose (L, \leq) is a poset with a $\neg\neg$ -stable order. For all $A \in \mathcal{P}(L)$ and $x \in L$, if x is the supremum of $\text{cl}_{\neg\neg} A$ then x is the supremum of A .*

Proof. By definition of $\text{cl}_{\neg\neg}$, $y \in \text{cl}_{\neg\neg} A$ is equivalent to $\neg\neg(y \in A)$. If \leq is $\neg\neg$ -stable then

$$\begin{aligned} \text{bound}(x, \text{cl}_{\neg\neg} A) &\iff \forall y \in L. (\neg\neg(y \in A) \implies y \leq x) \\ &\iff \forall y \in L. (y \in A \implies \neg\neg(y \leq x)) \\ &\iff \forall y \in L. (y \in A \implies y \leq x) \\ &\iff \text{bound}(x, A). \end{aligned}$$

Because $\text{cl}_{\neg\neg} A$ and A have the same upper bounds, if x is the supremum of one of them then it is the supremum of the other as well. \square

By a *chain* in a poset (L, \leq) we mean $C \in \mathcal{P}(L)$ such that

$$\forall x, y \in L. (x \in C \wedge y \in C \implies x \leq y \vee y \leq x).$$

The *object of chains in L* is defined as

$$\mathcal{C}(L) = \{C \in \mathcal{P}(L) \mid \forall x, y \in L. (x \in C \wedge y \in C \implies x \leq y \vee y \leq x)\}.$$

Proposition 3.2 *Every chain is discrete, i.e., $\mathcal{C}(L) \subseteq \mathcal{D}(L)$.*

Proof. Consider any $C \in \mathcal{C}(L)$ and $f : \nabla 2 \rightarrow L$ such that $\forall p \in \nabla 2. f(p) \in C$. We need to show that f is constant. Because C is a chain we have

$$\forall p, q \in \nabla 2. (f(p) \leq f(q) \vee f(q) \leq f(p)).$$

By a double application of Lemma 2.2 we obtain

$$(\forall p, q \in \nabla 2. f(p) \leq f(q)) \vee (\forall p, q \in \nabla 2. f(q) \leq f(p)).$$

Because \leq is antisymmetric, either of these two cases implies $f(p) = f(q)$ for all $p, q \in \nabla 2$, as required. \square

4 The poset $\nabla \omega_1$

Let (ω_1, \preceq) be the distributive lattice of countable ordinals in **Set**. This is not a chain-complete poset, but it is complete with respect to countable subsets. Let $\text{sup} : P_\omega(\omega_1) \rightarrow \omega_1$ be the supremum operator which maps a countable subset $A \subseteq \omega_1$ to its supremum.

The object $\nabla \omega_1$, ordered by $\nabla \preceq$, is a distributive lattice in **Eff**. One way to see this is to observe that ∇ preserves finite products, therefore it maps models of the equational theory of distributive lattices to models of the same theory. Moreover, ∇ also preserves the statement

$$\forall A \in P_\omega(S). \text{“sup}(A) \text{ is the supremum of } A\text{”}$$

because the statement is expressed in the negative fragment of logic $(\wedge, \implies, \forall)$, which is preserved by ∇ .

Proposition 4.1 *The poset $\nabla \omega_1$ is chain-complete in **Eff**.*

Proof. We claim that the supremum operator $\mathcal{C}(\nabla \omega_1) \rightarrow \nabla \omega_1$ is the composition

$$\mathcal{C}(\nabla \omega_1) \xrightarrow{\subseteq} \mathcal{D}(\nabla \omega_1) \xrightarrow{\text{cl}_{\neg \neg}} \nabla(P_\omega(\omega_1)) \xrightarrow{\nabla \text{sup}} \nabla \omega_1$$

The arrow marked by \subseteq comes from Lemma 3.2, while the one marked as $\text{cl}_{\neg\neg}$ is the factorization $\mathcal{D}(\nabla\omega_1) \rightarrow \nabla P_\omega(\omega_1)$ from Proposition 2.1.

We argue in the internal language of Eff . Consider a chain $C \in \mathcal{C}(\nabla\omega_1)$. Then $\text{cl}_{\neg\neg}C \in \nabla P_\omega(\omega_1)$, therefore $(\nabla \text{sup})(\text{cl}_{\neg\neg}C)$ is the supremum of $\text{cl}_{\neg\neg}C$. But since the order $\nabla \leq$ on $\nabla\omega_1$ is $\neg\neg$ -stable it is also the supremum of C by Lemma 3.1. \square

Corollary 4.2 *In the effective topos, there is a chain-complete distributive lattice with a monotone and progressive endomap which does not have a fixed point.*

Proof. The successor map $\text{succ} : \omega_1 \rightarrow \omega_1$ is monotone, progressive, and does not have a fixed point. The functor ∇ preserves these properties because they are all expressed in the negative fragment. Therefore, in the effective topos $\nabla\omega_1$ is a chain-complete distributive lattice and ∇succ is monotone, progressive and does not have a fixed point. \square

5 Discussion

An immediate consequence of Corollary 4.2 is that the following theorems *cannot* be proved constructively, i.e., in higher-order intuitionistic logic:

1. Tarski's Theorem [7] for chain-complete lattices: a monotone map on a chain-complete lattice has a fixed point.
2. Bourbaki-Witt theorem [1, 9]: a progressive map on a chain-complete poset has a fixed point above every point.

The theorems cannot be proved even if we assume Dependent Choice because it is valid in the effective topos.

Dito Pataraiia [6] proved constructively Tarski's fixed-point theorem for dpos . A natural question is whether perhaps the Bourbaki-Witt theorem can also be proved constructively for dpos . The following observation by France Dacar [2] shows that this is not possible because the Bourbaki-Witt theorems for chain-complete posets and dpos are constructively equivalent.

Theorem 5.1 (France Dacar) *The following are constructively equivalent:*

1. *Every progressive map on a chain-complete inhabited poset has a fixed point.*
2. *Every progressive map on a directed-complete inhabited poset has a fixed point.*

Proof. For this theorem we require chains to be inhabited.³ The direction from chain-complete posets to directed-complete ones is trivial because every directed-complete poset is chain-complete. To prove the converse, suppose (2) holds and let (P, \leq) be a chain-complete inhabited poset with a progressive map $f : P \rightarrow P$. The set C of inhabited chains in P , ordered by inclusion, is inhabited and closed under directed unions, therefore it is a dcpo. Define the map $F : C \rightarrow C$ by $F(A) = A \cup f(\sup(A))$. This is a progressive map on C , therefore by (2) it has a fixed point B . Now $f(\sup(B)) \in B$ and hence $f(\sup B) \leq \sup B$, which means that $\sup(B)$ is a fixed point of f . \square

In constructive mathematics the tradition is not to despair when a classical theorem turns out to be unprovable, but rather to find a constructively acceptable formulation and prove it. What that might be in the present case remains to be seen.

Acknowledgment. I thank France Dacar for inspiration and many useful bits of knowledge.

References

- [1] Nicolas Bourbaki. Sur le théorème de Zorn. *Archiv der Mathematik*, 2(6):434–437, November 1949.
- [2] France Dacar. Suprema of families of closure operators. Seminar for foundations of mathematics and theoretical computer science, November 2008. Faculty of Mathematics and Physics, University of Ljubljana, Slovenia.
- [3] J. Martin E. Hyland. The effective topos. In A.S. Troelstra and D. Van Dalen, editors, *The L.E.J. Brouwer Centenary Symposium*, pages 165–216. North Holland Publishing Company, 1982.
- [4] J. Martin E. Hyland, Edmund P. Robinson, and Giuseppe Rosolini. The discrete objects in the effective topos. *Proceedings of the London Mathematical Society*, 60:1–60, 1990.
- [5] F. William Lawvere. Diagonal arguments and cartesian closed categories. *Theory and Applications of Categories*, (15):1–13, 2006. Originally published in: *Lecture Notes in Mathematics*, No. 92, pages 134–135, 1969.
- [6] Dito Pataraiia. A constructive proof of Tarski’s fixed-point theorem for dcpos. 65th Peripatetic Seminar on Sheaves and Logic, November 1997.

³So far we could work with possibly uninhabited chains because the poset of interest $\nabla\omega_1$ has a least element.

- [7] Alfred Tarski. A lattice-theoretical fixpoint theorem and its applications. *Pacific Journal of Mathematics*, 5(2):285–309, 1955.
- [8] Jaap van Oosten. *Realizability: An Introduction to its Categorical Side*, volume 152 of *Studies in Logic and the Foundations of Mathematics*. Elsevier, 2008.
- [9] Ernst Witt. Beweisstudien zum Satz von M. Zorn. *Mathematische Nachrichten*, 4:434–438, 1951.

A The Effective Topos

We rely on [8] as a reference on the effective topos and give only a quick overview of the basic constructions here.

A.1 Definition of the effective topos

Recall that a *non-standard* predicate on a set X is a map $A : X \rightarrow P(\mathbb{N})$, where we think of $A(x)$ as the set of realizers (Gödel codes of programs) which witness the fact that x has the property A . The non-standard predicates on X form a Heyting prealgebra $P(\mathbb{N})^X$ with the partial order

$$A \leq B \iff \exists n \in \mathbb{N} . \forall x \in X . \forall m \in A(x) . \varphi_n(m) \downarrow \wedge \varphi_n(m) \in B(x),$$

where φ_n is the n -th partial recursive function and $\varphi_n(m) \downarrow$ means that $\varphi_n(m)$ is defined. In words, A entails B if there is a program that translates realizers for $A(x)$ to realizers for $B(x)$, uniformly in x . Predicates A and B are *equivalent* when $A \leq B$ and $B \leq A$. If we quotient $P(\mathbb{N})^X$ by the equivalence we obtain an honest Heyting algebra, but we do not do that.

Let $\langle -, - \rangle$ be a computable pairing function on the natural numbers \mathbb{N} , e.g., $\langle m, n \rangle = 2^m(2n + 1)$. The Heyting prealgebra structure of $P(\mathbb{N})^X$ is as follows:

$$\begin{aligned} \top(x) &= \mathbb{N} & (3) \\ \perp(x) &= \emptyset \\ (A \wedge B)(x) &= \{ \langle m, n \rangle \mid m \in A(x) \wedge n \in B(x) \} \\ (A \vee B)(x) &= \{ \langle 0, n \rangle \mid n \in A(x) \} \cup \{ \langle 1, n \rangle \mid n \in B(x) \} \\ (A \Rightarrow B)(x) &= \{ n \in \mathbb{N} \mid \forall m \in A(x) . \varphi_n(m) \downarrow \wedge \varphi_n(m) \in B(x) \}. \end{aligned}$$

We say that a non-standard predicate A is *valid* if $\top \leq A$, in which case we write $\models A$. The condition $\top \leq A$ is equivalent to requiring that $\bigcap_{x \in X} A(x)$ contains at least one number. Often a non-standard predicate is given as a map $x \mapsto \phi(x)$ where ϕ is an expression with a free variable x . In this case we abuse notation and write $\models \phi(x)$ instead of $\models \lambda x : X . \phi(x)$. In other words, free variables are to be implicitly abstracted over.

An object $X = (|X|, =_X)$ in the effective topos is a set $|X|$ with a non-standard *equality predicate* $=_X : |X| \times |X| \rightarrow P(\mathbb{N})$, which is required to be symmetric and transitive (where we write $[x =_X y]$ instead of $x =_X y$ for better readability):

$$\begin{aligned} \models [x =_X y] &\Rightarrow [y =_X x], && \text{(symmetric)} \\ \models [x =_X y] \wedge [y =_X z] &\Rightarrow [x =_X z]. && \text{(transitive)} \end{aligned}$$

Usually we write $\mathbf{E}_X(x)$ for $[x =_X x]$. Think of \mathbf{E}_X as an “existence predicate”, and $\mathbf{E}_X(x)$ as the set of realizers which witness the fact that x exists.

In the effective topos a morphism $F : X \rightarrow Y$ is represented by a non-standard functional relation $F : X \times Y \rightarrow P(\mathbb{N})$. More precisely, we require that

$$\begin{aligned} \models F(x, y) &\Rightarrow \mathbf{E}_X(x) \wedge \mathbf{E}_Y(y) && \text{(strict)} \\ \models [x' =_X x] \wedge F(x, y) \wedge [y =_Y y'] &\Rightarrow F(x', y') && \text{(extensional)} \\ \models F(x, y) \wedge F(x, y') &\Rightarrow [y =_X y'] && \text{(single-valued)} \\ \models \mathbf{E}_X(x) &\Rightarrow \bigcup_{y \in Y} \mathbf{E}_Y(y) \wedge F(x, y). && \text{(total)} \end{aligned}$$

Two such functional relations F, F' represent the same morphism when $F \leq F'$ and $F' \leq F$ in the Heyting prealgebra $P(\mathbb{N})^{X \times Y}$. Composition of $F : X \rightarrow Y$ and $G : Y \rightarrow Z$ is the functional relation $G \circ F$ given by

$$(G \circ F)(x, z) = \bigcup_{y \in Y} F(x, y) \wedge G(y, z).$$

The identity morphism $I : X \rightarrow X$ is represented by the relation $I(x, y) = [x =_X y]$.

The category \mathbf{Eff} is a topos. Let us give a description of powerobjects. If X is an object then the *powerobject* $\mathcal{P}(X)$ is the set $P(\mathbb{N})^{|X|}$ with the non-standard equality predicate

$$\begin{aligned} [A =_{\mathcal{P}(X)} B] &= (A \Rightarrow B) \wedge (B \Rightarrow A) \wedge \\ &\left(\bigcap_{x \in |X|} A(x) \Rightarrow \mathbf{E}_X(x) \right) \wedge \left(\bigcap_{x, y \in |X|} A(x) \wedge [x =_X y] \Rightarrow A(y) \right). \end{aligned}$$

The complicated part in the second line says that A is strict and extensional. If x and y are variables of type X and $\mathcal{P}(X)$, respectively, then the atomic predicate $x \in y$ is represented by the strict extensional predicate $E : |X| \times P(\mathbb{N})^{|X|} \rightarrow P(\mathbb{N})$ defined by $E(u, A) = \mathbf{E}_X(u) \wedge \mathbf{E}_{\mathcal{P}(X)}(A) \wedge A(u)$.

A.2 Interpretation of first-order logic in \mathbf{Eff}

The effective topos supports an interpretation of intuitionistic first-order logic, which we outline in this section. Each subobject of an object $X =$

$(|X|, =_X)$ is represented by a *strict extensional predicate*, which is a non-standard predicate $A : |X| \rightarrow P(\mathbb{N})$ that satisfies:

$$\begin{aligned} \models A(x) &\Rightarrow \mathbf{E}_X(x), && \text{(strict)} \\ \models A(x) \wedge [x =_X x'] &\Rightarrow A(x'). && \text{(extensional)} \end{aligned}$$

Such a predicate represents the subobject determined by the mono $I : Y \rightarrow X$ where $|Y| = |X|$, $[x =_Y y] = [x =_X y] \wedge A(x)$, and $I(x, y) = [x =_Y y]$. Strict predicates represent the same subobject precisely when they are equivalent as elements of the Heyting prealgebra $P(\mathbb{N})^X$.

The interpretation of first-order logic with equality in \mathbf{Eff} may be expressed in terms of strict extensional predicates and non-standard equality predicates. Suppose ϕ is a formula with a free variable x ranging over an object X .⁴ The interpretation of ϕ is the subobject of X represented by the non-standard predicate $\llbracket \phi \rrbracket : |X| \rightarrow P(\mathbb{N})$, defined inductively on the structure of ϕ as follows. The propositional connectives are interpreted by the Heyting prealgebra structure of non-standard predicates, cf. (3):

$$\begin{aligned} \llbracket \top \rrbracket &= \top \\ \llbracket \perp \rrbracket &= \perp \\ \llbracket \theta \wedge \psi \rrbracket &= \llbracket \theta \rrbracket \wedge \llbracket \psi \rrbracket \\ \llbracket \theta \vee \psi \rrbracket &= \llbracket \theta \rrbracket \vee \llbracket \psi \rrbracket \\ \llbracket \theta \Rightarrow \psi \rrbracket &= \llbracket \theta \rrbracket \Rightarrow \llbracket \psi \rrbracket. \end{aligned}$$

Suppose ψ is a formula with free variables x of type X and y of type Y , and let $A = \llbracket \psi \rrbracket : |X| \times |Y| \rightarrow P(\mathbb{N})$ be a strict extensional predicate which interprets ψ . Then the interpretation of the quantifiers is:

$$\begin{aligned} \llbracket \exists x \in X . \psi \rrbracket(y) &= \bigcup_{x \in |X|} \mathbf{E}_X(x) \wedge A(x, y), && (4) \\ \llbracket \forall x \in X . \psi \rrbracket(y) &= \bigcap_{x \in |X|} \mathbf{E}_X(x) \Rightarrow A(x, y). \end{aligned}$$

Suppose $f, g : X \rightarrow Y$ are morphisms represented by functional relations $F, G : |X| \times |Y| \rightarrow P(\mathbb{N})$, respectively. The atomic formula $f = g$, where x is a variable of type X , is interpreted as the subobject of X represented by the non-standard predicate $\llbracket f = g \rrbracket : |X| \rightarrow P(\mathbb{N})$, defined by

$$\llbracket f = g \rrbracket(x) = \bigcup_{y \in |Y|} F(x, y) \wedge G(x, y).$$

If other atomic predicates appear in a formula, their interpretation must be given in terms of corresponding strict extensional predicates.

⁴In the general case ϕ may contain free variables x_1, \dots, x_n ranging over objects X_1, \dots, X_n , respectively. Such a ϕ is interpreted as a subobject of $X_1 \times \dots \times X_n$. It is easy to work out the details once you have seen the case of a single variable.

A.3 The functor $\nabla : \mathbf{Set} \rightarrow \mathbf{Eff}$

The topos of sets \mathbf{Set} is (equivalent to) the topos of sheaves for the $\neg\neg$ -topology on \mathbf{Eff} . The direct image part of the inclusion $\mathbf{Set} \rightarrow \mathbf{Eff}$ is the functor $\nabla : \mathbf{Set} \rightarrow \mathbf{Eff}$ which maps a set S to the object $\nabla S = (S, =_{\nabla S})$ where

$$[x =_{\nabla S} y] = \begin{cases} \mathbb{N} & \text{if } x = y, \\ \emptyset & \text{if } x \neq y. \end{cases}$$

A map $f : S \rightarrow T$ is mapped to the morphism $\nabla f : \nabla S \rightarrow \nabla T$ represented by the functional relation

$$(\nabla f)(x, y) = [f(x) =_{\nabla T} y].$$

The inverse image part is the global sections functor $\Gamma : \mathbf{Eff} \rightarrow \mathbf{Set}$, defined as $\Gamma(X) = \mathbf{Eff}(\mathbf{1}, X)$. Concretely, a global point $\mathbf{1} \rightarrow X$ is represented by an element $x \in |X|$ such that $E_X(x) \neq \emptyset$. Two such $x, y \in |X|$ represent the same global point when $[x =_X y] \neq \emptyset$.

If S is a set then every element of ∇S exists uniformly, in the sense that $E_S(x) = \mathbb{N}$. Every map $S \rightarrow P(\mathbb{N})$ is strict and extensional with respect to $=_{\nabla S}$. These two observations allow us to simplify calculations involving ∇S . For example, the powerobject $\mathcal{P}(\nabla S)$ is the set $P(\mathbb{N})^S$ with the equality predicate simplified to $[A =_{\mathcal{P}(\nabla S)} B] = (A \Rightarrow B) \wedge (B \Rightarrow A)$. Similarly, the interpretation (4) of existential and universal quantifiers simplifies to

$$\begin{aligned} \llbracket \exists x \in \nabla S . \psi \rrbracket (y) &= \bigcup_{x \in S} A(x, y), \\ \llbracket \forall x \in \nabla S . \psi \rrbracket (y) &= \bigcap_{x \in S} A(x, y). \end{aligned}$$