Synthetic Computability

Andrej Bauer

Department of Mathematics and Physics
University of Ljubljana
Slovenia

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What is "synthetic" mathematics?

- Suppose we want to study mathematical structures forming a category $\mathcal{C}$, such as:
  - smooth manifolds and differentiable maps
  - topological spaces and continuous maps
  - computable sets and computable maps
- **Classical approach:** objects are *sets equipped with extra structure*, morphisms *preserve the structure*.
- **Synthetic approach:** embed $\mathcal{C}$ in a suitable *mathematical universe* $\mathcal{E}$ (a model of intuitionistic set theory) and view structures as *ordinary sets* and morphisms as *ordinary maps* inside $\mathcal{E}$.

A synthetic universe for computability theory

- M. Hyland's *effective topos* Eff is the mathematical universe suitable for computability theory.
- In Eff all objects and morphisms are equipped with computability structure.
- We need not know how Eff is built—we just use the logic and axioms which are valid in it.
- In the next lecture we will learn more about Eff.

External and internal view

Comparison of concepts as viewed by us (externally) and by mathematicians inside Eff (internally):

Symbol	**External**	**Internal**
$\mathbb{N}$	natural numbers	natural numbers
$\mathbb{R}$	*computable* reals	*all* reals
$f : \mathbb{N} \to \mathbb{N}$	*computable* map	*any* map
$e : \mathbb{N} \twoheadrightarrow A$	*computable* enumeration of A	*any* enumeration of A
$\{\text{true}, \text{false}\}$	truth values	decidable truth values
Ω	truth values of Eff	truth values
$\forall x$	*computably* for all x	for all x
$\exists x$	there exists *computable* x	there exists x
$P \lor \neg P$	decision procedure for P	P or not P

Related Work

- Friedman [1971], axiomatizes coding and universal functions
- Moschovakis [1971] & Fenstad [1974], axiomatize computations and subcomputations
- Hyland [1982], effective topos
- Richman [1984], an axiom for effective enumerability of partial functions, extended in Bridges & Richman [1987]
- We shall follow Richman [1984] in style, and borrow ideas from Rosolini [1986], Berger [1983], and Spreen [1998].

Outline

Intuitionistic logic

- We use *intuitionistic logic*, more precisely the internal language of a topos.
- What is the status of Law of Excluded Middle (LEM)?

$$\forall p \in \Omega \,.\, (p \vee \neg p)$$

"For every proposition p, p or not p."

  In intuitionistic mathematics it can only be used in special cases, when p is *decidable*.
- At this point we do not know whether all propositions are decidable, but later one of our axioms will falsify LEM.
- The status of the Axiom of Choice will be discussed later.

Basic sets and constructions

- Basic sets:

$$\emptyset, \quad \mathsf{1} = \{*\}, \quad \mathbb{N} = \{0, 1, 2, \ldots\}$$

- Set operations:

$$A \times B, \quad A + B, \quad B^A = A \to B, \quad \{x \in A \mid p(x)\}, \quad \mathcal{P}A$$

- We say that A is
  - *non-empty* if $\neg \forall\, x \in A \,.\, \bot$,
  - *inhabited* if $\exists\, x \in A \,.\, \top$.

Relations and functions

- A relation $R \subseteq A \times B$ is:
  - *single-valued* if $\langle x, y\rangle \in R \wedge \langle x, z\rangle \in R \implies y = z$,
  - *total* if $\forall\, x \in A \,.\, \exists\, y \in B \,.\, \langle x, y\rangle \in R$,
  - *functional* if it is single valued and total.
- Every $R \subseteq A \times B$ determines $f : A \to \mathcal{P}B$, and vice versa

  $$f(x) = \{y \in B \mid \langle x, y\rangle \in R\} \quad \text{and} \quad \langle x, y\rangle \in R \iff y \in f(x)$$

  We say that R is the *graph* of f.
- Relations as functions:
  - single-valued relations are *partial functions* $f : A \rightharpoonup B$,
  - total relations are *multi-valued functions* $f : A \rightrightarrows B$,
  - functional relations are just *functions* $f : A \to B$.

Axiom of Choice

- Axiom of Choice:

  Every $f : A \rightrightarrows B$ has a choice function $g : A \to B$ such that $g(x) \in f(x)$ for all $x \in A$.

  This we do not accept because it implies LEM.

- We accept *Number Choice*:

  Every $f : \mathbb{N} \rightrightarrows B$ has a choice function $g : \mathbb{N} \to B$.

- We also accept *Dependent Choice*:

  Given $x \in A$ and $h : A \rightrightarrows A$, there exists $g : \mathbb{N} \to A$ such that $g(0) = x$ and $g(n+1) \in h(g(n))$ for all $n \in \mathbb{N}$.

  This is a form of *simple recursion* for multi-valued functions.

Sets of truth values

- The set of truth values:

$$\Omega = \mathcal{P}\mathsf{1}$$
$$\text{truth } \top = \mathsf{1}, \quad \text{falsehood } \bot = \emptyset$$

- The set of *decidable* truth values:

$$\mathsf{2} = \{0, 1\} = \{p \in \Omega \mid p \vee \neg p\} \ ,$$

  where we write $1 = \top$ and $0 = \bot$.

- The set of *classical* truth values:

$$\Omega_{\neg\neg} = \{p \in \Omega \mid \neg\neg p = p\} \ .$$

- $\mathsf{2} \subseteq \Omega_{\neg\neg} \subseteq \Omega$.

Decidable and classical sets

- A subset $S \subseteq A$ is equivalently given by its characteristic map

$$\chi_S : A \to \Omega, \qquad \chi_S(x) = (x \in S).$$

- A subset $S \subseteq A$ is *decidable* if $\chi_S : A \to \mathsf{2}$, equivalently

$$\forall\, x \in A\,.\,(x \in S \lor x \notin S)\ .$$

- A subset $S \subseteq A$ is *classical* if $\chi_S : A \to \Omega_{\neg\neg}$, equivalently

$$\forall\, x \in A\,.\,(\neg(x \notin S) \implies x \in S)\ .$$

Enumerable & finite sets

- A is *finite* if there exist $n \in \mathbb{N}$ and a surjection

$$e : \{1, \ldots, n\} \twoheadrightarrow A,$$

  called a *listing* of A. An element may be listed more than once.

- A is *enumerable (countable)* if there exists a surjection

$$e : \mathbb{N} \twoheadrightarrow \mathsf{1} + A,$$

  called an *enumeration* of A. For inhabited A we may take $e : \mathbb{N} \twoheadrightarrow A$.

- A is *infinite* if there exists an injective $a : \mathbb{N} \rightarrowtail A$.

Outline

Lawvere → Cantor

Theorem (Lawvere)

If $e : A \twoheadrightarrow B^A$ is surjective then B has the fixed point property: for every $f : B \to B$ there is $x_0 \in B$ such that $f(x_0) = x_0$.

Proof.

Given $f : B \to B$, define $g(y) = f(e(y)(y))$. Because e is surjective there is $x \in A$ such that $e(x) = g$. Then $e(x)(x) = f(e(x)(x))$, so $x_0 = e(x)(x)$ is a fixed point of f. □

Corollary (Cantor)

There is no surjection $e : A \twoheadrightarrow \mathcal{P}A$.

Proof.

$\mathcal{P}A = \Omega^A$ and $\neg : \Omega \to \Omega$ does not have a fixed point. □

Non-enumerability of Cantor and Baire space

Are there any sets which are *not* enumerable?
Yes, for example $\mathcal{P}\mathbb{N}$, and also:

Corollary

$\mathsf{2}^{\mathbb{N}}$ *and* $\mathbb{N}^{\mathbb{N}}$ *are not enumerable.*

Proof.

$\mathsf{2}$ and $\mathbb{N}$ do not have the fixed-point property. □

We have proved our first synthetic theorem:

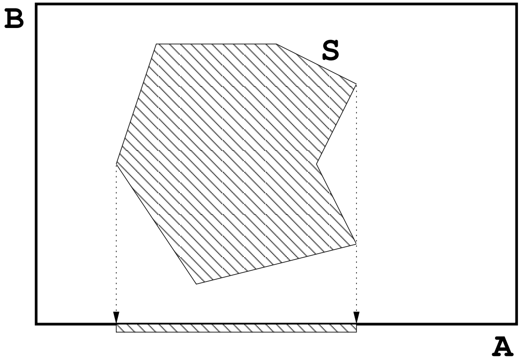
Theorem (external translation of above corollary)

The set of recursive sets and the set of total recursive functions cannot be computably enumerated.

Projection Theorem

Recall: the *projection* of $S \subseteq A \times B$ is the set

$$\{x \in A \mid \exists y \in B \,.\, \langle x, y\rangle \in S\}\ .$$



Projection Theorem

Theorem (Projection)

A subset of $\mathbb{N}$ is enumerable iff it is the projection of a decidable subset of $\mathbb{N} \times \mathbb{N}$.

Proof.

If A is enumerated by $e : \mathbb{N} \to 1 + A$ then A is the projection of the *graph* of e,

$$\{\langle m, n\rangle \in \mathbb{N} \times \mathbb{N} \mid m = e(n)\}.$$

If A is the projection of $B \subseteq \mathbb{N} \times \mathbb{N}$, define $e : \mathbb{N} \times \mathbb{N} \to 1 + A$ by

$$e\langle m, n\rangle = \texttt{if } \langle m, n\rangle \in B \texttt{ then } m \texttt{ else } \star \ .$$

□

Semidecidable sets

- A *semidecidable truth value* $p \in \Omega$ is one that is equivalent to
$$\exists\, n \in \mathbb{N} \,.\, d(n)$$
for some $d : \mathbb{N} \to \mathsf{2}$.
- The set of semidecidable truth values:
$$\Sigma = \{p \in \Omega \mid \exists\, d \in \mathsf{2}^{\mathbb{N}} \,.\, (p \iff \exists\, n \in \mathbb{N} \,.\, d(n))\} \,.$$
This is a *dominance*.
- $\mathsf{2} \subseteq \Sigma \subseteq \Omega$.
- A subset $S \subseteq A$ is *semidecidable* if $\chi_S : A \to \Sigma$.

Semidecidable subsets of $\mathbb{N}$

Theorem

The enumerable subsets of $\mathbb{N}$ are the semidecidable subsets of $\mathbb{N}$.

Proof.

An enumerable $A \subseteq \mathbb{N}$ is the projection of a decidable $B \subseteq \mathbb{N} \times \mathbb{N}$. Then $n \in A$ iff $\exists\, m \in \mathbb{N} \,.\, \langle n, m \rangle \in B$.
Conversely, if $A \in \Sigma^{\mathbb{N}}$, by Number Choice there is $d : \mathbb{N} \times \mathbb{N} \to \mathsf{2}$ such that $n \in A$ iff $\exists\, m \in \mathbb{N} \,.\, d(m, n)$. □

The enumerable subsets of $\mathbb{N}$:

$$\mathcal{E} = \Sigma^{\mathbb{N}} \,.$$

Note: at this point we do *not* know whether $\mathcal{E} = \mathcal{P}\mathbb{N}$.

The Single-Value Theorem

A *selection* for $R \subseteq A \times B$ is a partial map $f : A \rightharpoonup B$ such that, for every $x \in A$,

$$(\exists y \in B \,.\, R(x, y)) \implies f(x){\downarrow} \wedge R(x, f(x)) \;.$$

This is like a choice function, expect it only chooses when there is something to choose from.

Theorem (Single Value Theorem)

Every semidecidable relation $R \in \Sigma^{\mathbb{N} \times \mathbb{N}}$ has a Σ-partial selection.

Partial functions

- Given a single-valued $R \subseteq B$, the corresponding $f : A \to \mathcal{P}B$ always factors through

$$\widetilde{B} = \{S \in \mathcal{P}B \mid \forall\, x, y \in B \,.\, (x \in S \wedge y \in S \implies x = y)\}.$$

- Thus partial maps $f : A \rightharpoonup B$ are just ordinary maps $f : A \to \widetilde{B}$.
- Write $f(x)\downarrow$ when f is defined at x, i.e., $\exists\, y \in B \,.\, y \in f(x)$.

Σ-partial functions

When does a partial $f : \mathbb{N} \rightharpoonup \mathbb{N}$ have an enumerable graph?

Proposition

$f : \mathbb{N} \to \widetilde{\mathbb{N}}$ has an enumerable graph iff $f(n){\downarrow} \in \Sigma$ for all $n \in \mathbb{N}$.

Define the *lifting* operation

$$A_\perp = \left\{ S \in \widetilde{A} \;\middle|\; (\exists\, x \in A \,.\, x \in S) \in \Sigma \right\} .$$

For $f : A \to B$ define $f_\perp : A_\perp \to B_\perp$ to be

$$f_\perp(s) = \left\{ f(x) \;\middle|\; x \in s \right\} .$$

A Σ-*partial function* is a function $f : A \to B_\perp$.

Domains of Σ-partial functions

The *support* (a.k.a. *domain*) of $f : A \rightharpoonup B$ is $\{x \in A \mid f(x)\downarrow\}$.

Proposition

A subset is semidecidable iff it is the support of a Σ-partial function.

Proof.

A semidecidable subset $S \in \Sigma^A$ is the domain of its characteristic map $\chi_S : A \to \Sigma = 1_\perp$.
Conversely, if $f : A \to B_\perp$ is Σ-partial then its domain is the set $\{x \in A \mid f(x)\downarrow\}$, which is obviously semidecidable. □

Theorem (External translation)

A set is semidecidable iff it is the domain (support) of a partial computable map.

Outline

Axiom of Enumerability

Axiom (Enumerability)

There are enumerably many enumerable sets of numbers.

Let $\mathsf{W} : \mathbb{N} \twoheadrightarrow \mathcal{E}$ be an enumeration.

Proposition

Σ *and* $\mathcal{E}$ *have the fixed-point property.*

Proof.

By Lawvere, $\mathsf{W} : \mathbb{N} \twoheadrightarrow \mathcal{E} = \Sigma^{\mathbb{N}} \cong \Sigma^{\mathbb{N} \times \mathbb{N}} \cong \mathcal{E}^{\mathbb{N}}$. □

The Law of Excluded Middle Fails

The Law of Excluded Middle says $\mathsf{2} = \Omega$.

Corollary

The Law of Excluded Middle is false.

Proof.

Among the sets $\mathsf{2} \subseteq \Sigma \subseteq \Omega$ only the middle one has the fixed-point property, so $\mathsf{2} \neq \Sigma \neq \Omega$. □

Immune and Simple Sets

- A set is *immune* if it is neither finite nor infinite.
- A set is *simple* if it is open and its complement is immune.

Theorem

There exists an immune subset of $\mathbb{N}$.

Proof.

Following Post, consider $P = \{\langle m, n\rangle \in \mathbb{N} \times \mathbb{N} \mid n > 2m \wedge n \in W_m\}$, and let $f : \mathbb{N} \to \mathbb{N}_\perp$ be a selection for P. We claim that

$$S = \mathrm{im}(f) = \{n \in \mathbb{N} \mid \exists\, m \in \mathbb{N} \,.\, f(m) = n\}$$

is simple and $\mathbb{N} \setminus S$ immune. Because $f(m) > 2m$ the set $\mathbb{N} \setminus S$ cannot be finite.

For any infinite enumerable set $U \subseteq \mathbb{N} \setminus S$ with $U = \mathsf{W}_m$, we have $f(m)\downarrow$, $f(m) \in \mathsf{W}_m = U$, and $f(m) \in S$, hence U is not contained in $\mathbb{N} \setminus S$. □

Enumerability of $\mathbb{N} \to \mathbb{N}_\perp$

Proposition

$\mathbb{N} \to \mathbb{N}_\perp$ is enumerable.

Proof.

Let $V : \mathbb{N} \twoheadrightarrow \Sigma^{\mathbb{N}\times\mathbb{N}}$ be an enumeration. By Single-Value Theorem and Number Choice, there is $\varphi : \mathbb{N} \to (\mathbb{N} \to \mathbb{N}_\perp)$ such that φ_n is a selection of V_n. The map φ is surjective, as every $f : \mathbb{N} \to \mathbb{N}_\perp$ is the only selection of its graph. □

Corollary (Formal Church's Thesis)

$\mathbb{N}^\mathbb{N}$ is sub-enumerable (because $\mathbb{N}^\mathbb{N} \subseteq \mathbb{N}_\perp^\mathbb{N}$).

In other words, $\forall f \in \mathbb{N}^\mathbb{N} \,.\, \exists n \in \mathbb{N} \,.\, f = \varphi_n$.

End of Part I

Walk around and rest your brain for 10 minutes.

Outline

Markov Principle

- If a binary sequence $a \in \mathbf{2}^{\mathbb{N}}$ is not constantly 0, does it contain a 1?
- For $p \in \Sigma$, does $p \neq \bot$ imply $p = \top$?
- Is $\Sigma \subseteq \Omega_{\neg\neg}$?

Axiom (Markov Principle)

A binary sequence which is not constantly 0 *contains a* 1.

Post's Theorem

Theorem

For all $p \in \Omega$*,*

$$p \in \mathsf{2} \iff p \in \Sigma \wedge \neg p \in \Sigma \,.$$

Proof.

- $\Rightarrow$ If $p \in \mathsf{2}$ then $\neg p \in \mathsf{2}$, therefore $p, \neg p \in \mathsf{2} \subseteq \Sigma$.
- $\Leftarrow$ If $p \in \Sigma$ and $\neg p \in \Sigma$ then $p \vee \neg p \in \Sigma \subseteq \Omega_{\neg\neg}$, therefore

$$p \vee \neg p = \neg\neg(p \vee \neg p) = \neg(\neg p \wedge \neg\neg p) = \neg\bot = \top \,,$$

as required.

□

Phoa's principle

What does $\Sigma \to \Sigma$ look like?

Theorem (Phoa's Principle)

For every $f : \Sigma \to \Sigma$ and $x \in \Sigma$,

$$f(x) = (f(\bot) \vee x) \wedge f(\top) \; .$$

The proof uses Enumeration axiom and Markov Principle. The principle says that $\Sigma \to \Sigma$ is a retract of $\Sigma \times \Sigma$ with

- section: $f \mapsto \langle f(\bot), f(\top) \rangle$
- retraction: $(u, v) \mapsto \lambda x : \Sigma \, . \, (u \vee x) \wedge v$

A consequence is monotonicity of $f : \Sigma \to \Sigma$: if $x \leq y$ then

$$f(x) = (f(\bot) \vee x) \wedge f(\top) \leq (f(\bot) \vee y) \wedge f(\top) = f(y) \; .$$

Outline

The Topological View

- The topological view:

$$\text{semidecidable subsets} = \text{open subsets}\ .$$

- Σ is the *Sierpinski space*: the space on two points $\bot, \top$ with $\{\top\}$ open and $\{\bot\}$ closed.
- The *topology of* A is Σ^A.
- "All functions are continuous."

  Given any $f : A \to B$ and $U \in \Sigma^B$, the set $f^{-1}(U)$ is open because it is classified by $U \circ f : A \to \Sigma$.

Topological Exterior and Creative Sets

- The *exterior* of an open set is the largest open set disjoint from it.
- An open set $U \in \Sigma^A$ is *creative* if it is without exterior: every $V \in \Sigma^A$ disjoint from U can be enlarged and still be disjoint from U.

Theorem

There exists a creative subset of $\mathbb{N}$.

Proof.

The familiar $K = \{n \in \mathbb{N} \mid n \in \mathsf{W}_n\}$ is creative. Given any $V \in \mathcal{E}$ with $V = \mathsf{W}_k$ and $K \cap V = \emptyset$, we have $k \notin V$ and $k \notin K$, so $V' = V \cup \{k\}$ is larger and still disjoint from K. □

The generic convergent sequence

- The one-point compactification of $\mathbb{N}$ is

$$\mathbb{N}^+ = \{a : \mathbb{N} \to \mathsf{2} \mid \forall n \in \mathbb{N} \,.\, a_n \leq a_{n+1}\} \,.$$

- A natural number n is represented by

$$\underbrace{0, 0, \ldots, 0}_{n}, 1, 1, \ldots$$

- Infinity ∞ corresponds to $0, 0, 0, \ldots$
- Σ is a quotient of $\mathbb{N}^+$ by $q : \mathbb{N}^+ \twoheadrightarrow \Sigma$,

$$q(a) = (a < \infty) = (\exists\, n \in \mathbb{N} \,.\, a_n = 1) \,.$$

The topology of $\mathbb{N}^+$

Theorem

Given $U : \mathbb{N}^+ \to \Sigma$, if $\infty \in U$ then $n \in U$ for some $n \in \mathbb{N}$.

Proof.

By Markov principle, it suffices to show that $\forall n \in \mathbb{N} \,.\, n \notin U$ implies $\infty \notin U$. Suppose $U : \mathbb{N}^+ \to \Sigma$ such that $\forall n \in \mathbb{N} \,.\, n \notin U$. Define a map $f : \Sigma \to \Sigma$ by $f(q(a)) = U(a)$. By monotonicity of f,

$$\bot \leq U(\infty) = f(\bot) \leq f(\top) = \bot \,.$$

□

The topology of an ω-cpo

A *ω-cpo* is a poset $(P, \leq)$ in which increasing chains have suprema.

Theorem

An open subset $U : P \to \Sigma$ is

- *upward closed: $x \in U \wedge x \leq y \implies y \in U$*
- *inaccessible by chains: given a chain $a : \mathbb{N} \to P$, if $\bigvee_k a_k \in U$ then $a_k \in U$ for some $k \in \mathbb{N}$.*

Proof.

(a) given $x \in U$ and $x \leq y$, define $f : \mathbb{N}^+ \to P$ by

$$f(u) = \bigvee_{k \in \mathbb{N}} \texttt{if } k < u \texttt{ then } x \texttt{ else } y\,.$$

Then $x = f(\infty) \in U$ hence for some $u < \infty$ we have $y = f(u) \in U$.
(b) Similarly, consider $f(u) = \bigvee_{k \in \mathbb{N}} a_{\min(k,u)}$. □

The Rice-Shapiro Theorem

- A *base* for an ω-cpo $(P, \leq)$ is an enumerable $B \subseteq P$ such that
  - for all $b \in B$ and $x \in P$ we have $(b \leq x) \in \Sigma$,
  - every $x \in P$ is the supremum of a chain of basic elements.

  Each basic $b \in B$ determines a *basic open*
  $\uparrow b = \{x \in P \mid b \leq x\}$.
- Example: a base for $\Sigma^{\mathbb{N}}$ is the family of finite subsets of $\mathbb{N}$.

Theorem (Rice-Shapiro)

In an ω-cpo with a base every open is the union of basic opens.

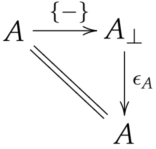
Proof.

$U : P \to \Sigma$ is the union of $\{\uparrow b \mid b \in U\}$. □

Outline

Focal sets

- A *focal set* is a set A together with a map $\epsilon_A : A_\perp \to A$ such that $\epsilon_A(\{x\}) = x$ for all $x \in A$:



  The *focus* of A is $\perp_A = \epsilon_A(\perp)$.
- A lifted set $A_\perp$ is always focal (because lifting is a monad whose unit is $\{-\}$).

Enumerable focal sets

- Enumerable focal sets, known as *Eršov complete sets*, have good properties.
- A *flat domain* $A_\perp$ is focal. It is enumerable if A is decidable and enumerable.
- If A is enumerable and focal then so is $A^{\mathbb{N}}$:

$$\mathbb{N} \xrightarrow{\varphi} \mathbb{N}_\perp^{\mathbb{N}} \xrightarrow{e_\perp^{\mathbb{N}}} A_\perp^{\mathbb{N}} \xrightarrow{\epsilon_A^{\mathbb{N}}} A^{\mathbb{N}}$$

- Some enumerable focal sets are

$$\Sigma^{\mathbb{N}}, \quad \mathsf{2}_\perp^{\mathbb{N}}, \quad \mathbb{N}_\perp^{\mathbb{N}}\ .$$

Recursion Theorem

Theorem (Recursion Theorem)

If $A^{\mathbb{N}}$ is enumerable then every $f : A \rightrightarrows A$ has a fixed point, i.e., $x \in A$ such that $x \in f(x)$.

Proof.

Let $\ell : \mathbb{N} \to A^{\mathbb{N}}$ be an enumeration. Then $e : \mathbb{N} \to A$ defined by $e(k) = \ell(k)(k)$ is onto as well. Let $h : \mathbb{N} \to A$ be a choice map such that $h(n) \in f(e(n))$ for all $n \in \mathbb{N}$. There is $j \in \mathbb{N}$ such that $\ell(j) = h$, from which we get a fixed point $e(j) = \ell(j)(j) = h(j) \in f(e(j))$. □

Note: The theorem requires *no* synthetic axioms, but we need the Axiom of Enumerability to find interesting examples of such A, e.g., enumerable focal sets.

Classical Recursion Theorem

Corollary (Classical Recursion Theorem)

For every $f : \mathbb{N} \to \mathbb{N}$ *there is* $n \in \mathbb{N}$ *such that* $\varphi_{f(n)} = \varphi_n$.

Proof.

In Recursion Theorem, take the enumerable focal set $A = \mathbb{N}_\bot^\mathbb{N}$ and the multi-valued function

$$F(g) = \{h \in \mathbb{N}_\bot^\mathbb{N} \mid \exists\, n \in \mathbb{N} \,.\, g = \varphi_n \wedge h = \varphi_{f(n)}\} \,.$$

There is g such that $g \in F(g)$. Thus there exists $n \in \mathbb{N}$ such that $\varphi_n = g = h = \varphi_{f(n)}$. □

Outline

Plotkin's Domain $\mathsf{2}_\perp^{\mathbb{N}}$

- In a *partially ordered set* $(P, \leq)$ we say that x and y are *incomparable* if $x \not\leq y$ and $y \not\leq x$.
- Must there always be a maximal element above an element of a poset?
- The set of Σ-partial binary functions $\mathbb{N} \to \mathsf{2}_\perp$ is a partially ordered:

  $$f \leq g \iff \forall\, n \in \mathbb{N} \,.\, f(n) \subseteq g(n)\ .$$

  This is *Plotkin's universal domain*.

Inseparable sets

Theorem

There exists an element of $\mathbb{N} \to \mathbf{2}_\perp$ *that is inconsistent with every maximal element.*

Proof.

Because $\mathbf{2}_\perp$ is focal and enumerable, $\mathbf{2}_\perp^{\mathbb{N}}$ is as well. Let $\psi : \mathbb{N} \twoheadrightarrow \mathbf{2}_\perp^{\mathbb{N}}$ be an enumeration, and let $t : \mathbf{2}_\perp \to \mathbf{2}_\perp$ be the isomorphism $t(x) = \neg_\perp x$ which exchanges 0 and 1, and fixes $\perp$. Consider $a \in \mathbf{2}_\perp^{\mathbb{N}}$ defined by $a(n) = t(\psi_n(n))$. If $b \in \mathbf{2}_\perp^{\mathbb{N}}$ is maximal with $b = \psi_k$, then $a(k) = \neg\psi_k(k) = \neg b(k)$. Because $a(k)$ and $b(k)$ are both total and different they are inconsistent. Hence a and b are inconsistent. □

Conclusion

- The theme: we should look for *elegant* presentations of structures we study. They can lead to new intuitions (and destroy old ones).
- These slides, and more, at `math.andrej.com`.

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