The Dedekind Reals in Abstract Stone Duality

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Abstract

Abstract Stone Duality (ASD) is a direct axiomatisation of the spaces and maps in general topology, whereas the traditional and all other contemporary approaches treat spaces as sets (types, objects of a topos) with additional structure.

ASD is presented as a λ-calculus, of which this paper provides a self-contained summary, the foundational background having been investigated in earlier work. This reveals the computational content of various topological notions and suggests how to calculate with them.

Here we take the first steps in real analysis within ASD, namely the construction of the real line using two-sided Dedekind cuts. Further topics, such as the Intermediate Value Theorem, are presented in a separate paper that builds on this one.

We prove that the closed interval is compact and overt. These concepts are defined in ASD using quantifiers, for which we give programs. These are related to optimisation in Moore's Interval Analysis. We also compare our definitions of the arithmetic operations with his.

The final section compares ASD with other systems of constructive and computable topology and analysis, in particular the failure of compactness in Recursive Analysis.

1 Introduction

The title of Richard Dedekind's paper [Ded72] begins with the word "Stetigkeit" (continuity). Irrational numbers are secondary to this fundamental notion — his construction gives us access to them only *via* their rational approximations. He also stresses the importance of geometrical intuition. In other words, the real line is not a naked *set* of Dedekind cuts, dressed by later mathematicians in an outfit of so-called "open" subsets, but has a *topology* right from its conception.

Dedekind cited square roots as an example of the way in which we use continuity to enrich the rational numbers, but even they — being defined by division — presuppose an inequality relation that we shall come to regard as topology. So let us first write down what it is that we are trying to construct. We include compactness of the closed interval because it is one of the most important properties that real analysts use. As we shall see, this is not just a theorem that we prove in passing, but a hotly debated issue in the foundations of analysis.

Definition 1.1 An object R is a ***Dedekind real line***[1] if
(a) it is an overt space (Proposition 8.1);
(b) it is Hausdorff, with an inequality or apartness relation, $\neq$ (Corollary 6.14);
(c) the closed interval $[0,1]$ is compact (Theorem 11.11);
(d) R is a field, where x^{-1} is defined iff $x \neq 0$ (Theorem 10.7);
(e) it has a total order, *i.e.* $(x \neq y) \Leftrightarrow (x < y) \vee (y < x)$ (Corollary 6.14);
(f) it is Dedekind complete, where the two halves of the cut are open (Theorem 8.7);
(g) and Archimedean, *i.e.* $p, q : R \vdash q > 0 \Rightarrow \exists n{:}\mathbb{Z}.\ q(n-1) < p < q(n+1)$, (Axiom 9.1).

The familiar arithmetical operations $+$, $-$ are $\times$ are of course computable algebraic structure on R, as are division and the (strict) relations $<$, $>$ and $\neq$ when we introduce suitable types for

[1] In this paper we shall use R for the object under construction, and $\mathbb{R}$ for that in classical or other forms of analysis.

their arguments and results. The topological properties of overtness and compactness are related to the logical quantifiers $\exists$ and $\forall$, which we shall come to see as additional computable structure.

Remark 1.2 Topologically, the new property of overtness is analogous to compactness, in that it is related to open subspaces in the way that compactness is to closed ones. As far as this paper is concerned, overtness is simply part of the axiomatic structure, but we shall show in following work [J] that it explains the situations in which equations $fx = 0$ for $f : \mathbb{R} \to \mathbb{R}$ can or cannot be solved. Computationally, overtness then provides a generic way of solving equations, when this is possible. The reason why you haven't heard of overt (sub)spaces before is that classical topology makes *all* spaces overt by *force majeure*, without providing the computational evidence.

Remark 1.3 All of this is done in a new system of constructive topology called Abstract Stone Duality. ASD generalises Dedekind's topological conception of the real line: in it, the topology is an inherent and unalienable part of a space, which is not a set to which open subsets have been added as an afterthought.

ASD has both a classical interpretation in terms of classical topological spaces and continuous maps, and a computable one involving datatypes and programs. Here and in [J] we focus on the computable aspects of ASD in real analysis.

Any proof carried out within the ASD calculus yields a program, expressed as an ASD term. In particular, our proofs of compactness and overtness of closed intervals provide programs for computing quantifiers of the form $\forall x{:}[d, u]$ and $\exists x{:}[d, u]$ respectively. These are general and powerful higher-order functions from which many useful computations in real analysis can be derived.

Remark 1.4 In Section 2 we describe the general aims and methods of ASD, and Section 3 explains the classical idea behind our construction of the real line, presenting it from the point of view of interval analysis and optimisation. Sections 4 and 5 contain a short survey of the calculus, which will also be useful for reference in connection with other applications of ASD besides analysis.

Turning from the general topics to the construction, Sections 3, 6–8 develop Dedekind cuts and Sections 9 and 10 consider their arithmetic, in which the formula for multiplication seems to be novel. Section 11 proves that the closed interval is compact and overt.

Finally, Section 12 compares ours with other schools of thought. In particular, we contrast compactness of the closed interval here with its pathological properties in Recursive Analysis, and comment on the status of ASD from the point of view of a constructivist in the tradition of Errett Bishop.

Remark 1.5 Some readers may think that the world does not need yet another construction of the real numbers. We emphasise that we are not just aiming to do this, but to lay the foundation for a new theory of real analysis and its computational implementation.

In the paper that follows [J], overt subspaces will replace totally bounded and located ones in constructive analysis, just as compact subspaces replaced closed and bounded ones in conceptually formulated classical analysis. The basic principles of analysis on the real line are studied, in particular convergence of Cauchy sequences, maxima of compact overt subspaces and connectedness. With the benefit of the Heine–Borel definition of compactness, we can developed these ideas in a *topological* style, in contrast of the *metrical* one that Bishop and others have used. Finally, the intermediate value theorem is proved — computably, as always in ASD!

2 Topology as lambda-calculus

Abstract Stone Duality is a direct axiomatisation of topology whose aim is to integrate it with computability theory and denotational semantics. The basic building blocks — spaces and maps — are taken as fundamental, rather than being manufactured from sets with extra structure. Nor do we suppose that the spaces are special objects inside a larger universe such as a topos, an approach taken by Synthetic Differential Geometry [Koc81, Law80] and Synthetic Domain

Theory [Hyl91, Ros86, Tay91]. Also, whilst the calculus of ASD is essentially λ-calculus with (simple) type theory, we don't identify types with sets or propositions, as is done in Martin-Löf's type theory.

Remark 2.1 In ASD there are spaces and maps. There are three basic spaces: the one-point space $\mathbf{1}$, the space of natural numbers $\mathbb{N}$ and the Sierpiński space Σ, which are axiomatised in terms of their universal properties. (Recall that, classically, the Sierpiński space has one open and one closed point.) We can form products of spaces, $X \times Y$, and exponentials of the form Σ^X, but *not* arbitrary Y^X. The theory also provides certain "Σ-split" subspaces, which we explain with reference to the real line in Sections 3, 5 and 7.

All maps between these spaces are continuous, not as a theorem but by definition — the calculus simply never introduces discontinuous functions. Maps are defined by a (form of) λ-calculus, so we sometimes refer to spaces as *types*.

Statements in the theory are expressed as equations between terms. Since not every formula of (for example) the predicate calculus is of this form, the theory imposes restrictions on what can be said within it. However, the terms of type Σ and Σ^X do behave very much like propositions and predicates on X, respectively. We can form conjunctions and disjunctions of such terms, but not implications or negations — these become equations *between* terms. In some cases there are also operators $\exists_X : \Sigma^X \to \Sigma$ and $\forall_X : \Sigma^X \to \Sigma$ that satisfy the same formal properties as the quantifiers (*cf.* Remark 4.15). When $\forall_X$ exists, X is said to be *compact*, and when $\exists_X$ exists, X is called *overt*.

Thus we can express various properties of topological spaces "in logical form", provided that we choose the right formulation. But at the end of the day it is important to remember that these are still just equations between terms. We must be cautious before interpreting terms of ASD as logical statements about sets of points, or conversely introducing such assertions into our calculus.

Remark 2.2 It may be said that we are making a rod for our own backs by restricting ourselves to a rather weak calculus — a criticism that puts us in good company with constructive mathematicians, who often face the same lack of understanding. But the ASD calculus *is* the calculus of spaces and maps. If some feature is missing from it, this is not because of our asceticism, but because spaces and maps do not possess it. The justification of this claim is that, starting from the axioms of ASD, we may reconstruct the categories of computably based locally compact locales [G] and of general locales over an elementary topos [H].

In such a weak calculus, it will not surprise you to hear that, as a rule, it is very difficult to know how to say anything at all. But experience has shown something very surprising — once we have *some* way of expressing an idea, it usually turns out to be the *right* way. In contrast, stronger calculi, *i.e.* those that take advantage of the logic of sets, type theory or a topos, offer numerous candidates for formulating an idea, but these then often lead to distracting counterexamples.

Definition 2.3 In the ***classical interpretation*** of ASD, a type X is a topological space and the exponential Σ^X is the topology (lattice of open subspaces) of X, regarded not as a set but as another topological space, equipped with the Scott topology.

Recall that in the ***Scott topology*** on Σ^X, a family $\mathcal{U} \subset \Sigma^X$ of open subspaces of X is open if it is "inaccessible by directed unions" — if $(\bigcup_{i\in I} U_i) \in \mathcal{U}$ then already $(\bigcup_{i\in F} U_i) \in \mathcal{U}$ for some finite subset $F \subset I$. This topology on any complete lattice (or poset with directed joins) is very familiar in domain theory and the study of continuous lattices and locally compact spaces [GHK$^+$80]. It is a (in fact, the most important) special case of the compact–open topology on general function-spaces that has been known at least since 1945.

A term in our calculus of type Σ^X with one parameter, of type Σ^Y, is an abstract morphism $\Sigma^Y \to \Sigma^X$. Its concrete classical interpretation is a ***Scott-continuous*** function that assigns an open subspace of X to each open subspace of Y in a way that preserves directed unions. Compare this with the inverse image operation f^* (which we call Σ^f or $\psi : \Sigma^Y \vdash \lambda x.\, \psi(fx) : \Sigma^X$) of a continuous function $f : X \to Y$, which also preserves *finitary* unions and intersections.

Despite the apparently classical features of the calculus in Section 4, this interpretation is also valid for *intuitionistic* locally compact locales (**LKLoc**) over any elementary topos.

Remark 2.4 ASD is also valid for "computably based" locally compact locales, and is in fact complete for them. That is, its term model is equivalent to a category of sober spaces that are equipped with computable structure expressing local compactness, and continuous functions that are "tracked" by programs acting on bases [G].

However, computation is a natural part of the calculus. There is no need to bolt a clumsy, old-fashioned theory of recursion on to the front of it. Domain theory can be developed within ASD [F], and then programming languages such as Plotkin's PCF can be translated into this, using the denotational semantics in [Plo77]. Conversely, the topological features of the calculus can be "normalised out" of its terms, leaving PROLOG-like programs [A, §11], or, more precisely, parallel λ-PROLOG programs.

The computational idea behind PROLOG is that a predicate in a certain restricted logical language can be interpreted as a program whose objective is to prove the predicate, and report instantiations for its free and existentially bound variables. If the predicate is false, the program either aborts or never halts. The basic process whereby this is done is unification, in which variables are assigned values according to the constraints imposed by the rest of the program, *i.e.* the values that they must have if the program is ever to terminate with a proof of the original predicate.

Remark 2.5 This gives a constructive meaning to the existential quantifier over an overt type, *i.e.* a ***choice principle***. However, this is weaker than that found in other constructive foundational systems, because for us the free variables of the predicate must also be of overt discrete Hausdorff type. That is, they must be either natural numbers or something very similar, such as rationals. *They cannot be real numbers, functions or predicates.* (This is topologically significant, because it constrains the sense in which the closed interval has the Heine–Borel "finite sub-cover" property in Section 11.)

This choice principle is the pure mathematical interpretation that we extract from the evaluation strategy of (parallel λ-) PROLOG. As far as actual execution is concerned, if a program has *any* free variables then it is not sufficiently well defined to be run. However, the strategy can be treated as deterministic when the free variables are numerical. Hence we can regard its result as a *function* of its inputs, the graph of this function being contained in the relation defined by the predicate. If the predicate is provably true for all numerical inputs *and the evaluation strategy is parallel and fair* then the function is total.

Remark 2.6 Returning to topology, each term of type Σ denotes an *open* subspace, namely the inverse image of the open point $\top \in \Sigma$. But by considering the inverse image of the closed point $\bot \in \Sigma$ instead, the same term also corresponds to a *closed* subspace. There is thus a bijection between open and closed subspaces, but it arises from their common classifiers, *not* by complementation. Indeed, *there is no such thing as complementation, as there is no underlying theory of sets* (or negation).

This extensional correspondence between terms and both open and closed subspaces gives rise to Axiom 4.9, the ***Phoa principle***. This axiom makes the difference between a λ-*notation* for topology that must still refer to set theory to prove its theorems, and a *calculus* that can prove theorems for itself, indeed in a computational setting.

Remark 2.7 The lattice duality between open and closed things (discrete/Hausdorff, overt/compact, open/proper) runs right through the theory. In terms of lattices, it comes as a result of moving the directed joins into the background (using general Scott continuity), so that we postulate *finitary* meets and joins in Σ, instead of the usual asymmetrical infinitary joins and finitary meets. This has the pleasing result that the treatment is often completely symmetrical. Whilst this cannot, of course, always happen, it does so so often that habitually transcribing formulae into their duals has turned out to be a very fruitful methodological principle. (If you look carefully, you will see that it has even been applied within this paper.)

On the other hand, constructive or intuitionistic logicians may feel uncomfortable using this calculus because, as you will see in Section 4, the Phoa principle gives it many features that look like classical logic. The reason for this is that it is a higher order logic of *closed* subspaces, just as much as it is of *open* ones. When it is functioning in its "closed" role, we are essentially using the symbols the wrong way round: for example, $\vee$ denotes intersection.

The discomfort will be particularly acute in this paper, as it deals with a Hausdorff space, where the diagonal $R \subset R \times R$ is closed (Definition 4.12). This means that we can discuss equality of two real numbers, $a, b : R$, using a predicate of type Σ. But since the logic of closed predicates is upside down, the (open) predicate is $a \neq_R b$, and for equality we have to write the equation $(a \neq b) \Leftrightarrow \bot$ between terms of type Σ (or $\neg(a \neq b)$ if you like), whereas we can say $(n = m) \Leftrightarrow \top$ about integers.

We are not asserting a doubly negated equality, because $\neq$ is not negated equality, and there is no negation of predicates. The equational statements $(-) \Leftrightarrow \top$ and $(-) \Leftrightarrow \bot$ merely say that the parameters of the expression $(-)$ belong respectively to certain open or closed subspaces, which are of the same status in the logic.

Of course, the same de Morgan-style duality extends to the quantifiers, $\forall$ in a compact (sub)space and $\exists$ in an overt one. This is another reason why they don't have the same meaning as in other constructive systems of topology and proof theory. As we saw, $\exists$ has an associated choice principle, but it is only $\mathbb{N}$–$\mathbb{N}$ for Σ^0_1-predicates. Whilst $\exists$ is true more often in ASD, $\forall$ is true less often, as we shall explain in Section 12.

3 Cuts and intervals

Turning from the general principles of ASD to the specific construction of the real line, we make use of some ideas from interval analysis, as well as the basic idea of Dedekind cuts.

Remark 3.1 Dedekind represented each real number $a \in \mathbb{R}$ as a pair of sets of rationals, $\{d \mid d \leq a\}$ and $\{u \mid a < u\}$. From a constructive point of view, it is preferable to use the strict inequality in both cases, thereby omitting a itself if it's rational, so we have

$$D_a \;\equiv\; \{d \mid d < a\} \quad \text{and} \quad U_a \;\equiv\; \{u \mid a < u\}.$$

These are disjoint inhabited open subsets that "touch" in the sense that

$$d < u \;\Rightarrow\; d \in D \;\vee\; u \in U,$$

although we shall call this condition *locatedness*. The idea that a set of *rationals* is open can be expressed order-theoretically, using a condition that we shall call *roundedness* (Definition 6.8).

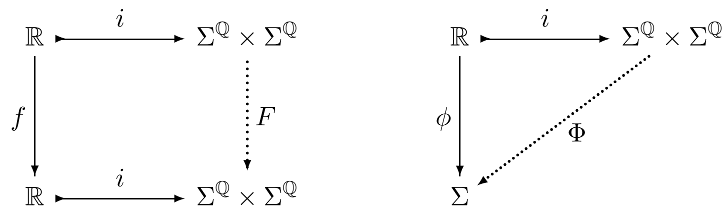
In ASD we represent the open subspaces D and U by λ-terms $\delta, \upsilon : \Sigma^{\mathbb{Q}}$. (Conventionally, we use Greek letters for terms of this type.) In particular, D_a and U_a become

$$\delta_a \;\equiv\; \lambda d.\, d < a \quad \text{and} \quad \upsilon_a \;\equiv\; \lambda u.\, a < u.$$

Hence the line itself is a subspace $i : \mathbb{R} \subset \Sigma^{\mathbb{Q}} \times \Sigma^{\mathbb{Q}}$. Since rational numbers are easily encoded, *e.g.* as pairs of integers, real numbers can therefore be represented as pairs of λ-terms.

Remark 3.2 We want to use this representation to compute with real numbers, and in the first instance to do arithmetic with them. Dedekind indicated how this can be done, defining operations on cuts. But there is a difference between his objective of providing a rigorous foundation for differential and integral calculus, and ours of getting a machine to compute with real numbers for us. He only had to define and justify the operations on legitimate cuts (those that are disjoint and located), whereas our machine will do *something* with *any* pair of λ-terms that we give it, even if it's only to print an error message.

It's reasonable to suppose that any program F that is intended to compute a function $f : \mathbb{R} \to \mathbb{R}$ using Dedekind cuts will actually take *any* pair of λ-terms and return another pair. We say that the program is *correct* for this task if, when it is given the pair (δ_a, υ_a) that represents a number a, it returns the pair $(\delta_{fa}, \upsilon_{fa})$ that represents the value $f(a)$ of the function at that input. In other words, the square on the left commutes:



Remark 3.3 The square on the right illustrates the similar situation of an open subspace of $\mathbb{R}$, classified by a continuous function $\phi : \mathbb{R} \to \Sigma$. Again we expect this to be the restriction to $\mathbb{R}$ of an open subspace Φ of $\Sigma^{\mathbb{Q}} \times \Sigma^{\mathbb{Q}}$. In other words, $\mathbb{R}$ should carry the ***subspace topology*** inherited from the ambient space. As we have already explained, $\Sigma^{\mathbb{Q}}$ itself carries the Scott topology. This extension of open subspaces is actually the more fundamental situation. We can use it to extend functions by defining

$$F(\delta, \upsilon) \;\equiv\; \big(\lambda d.\ \Phi_d(\delta, \upsilon),\ \lambda u.\ \Psi_u(\delta, \upsilon)\big),$$

where Φ_d and Ψ_u are the extensions of $\phi_d \equiv \lambda x.\, d < fx$ and $\psi_u \equiv \lambda x.\, fx < u$ respectively.

Definition 3.4 In practice, the most useful generalisation is to drop the locatedness condition, still requiring D and U to be disjoint open sets that extend towards $-\infty$ and $+\infty$ respectively. Instead of touching, and so representing a single real number $a \in \mathbb{R}$, they correspond to a *closed interval* $[d, u] \equiv \mathbb{R} \backslash (D \cup U)$, where we sometimes also allow $d \equiv -\infty$ and $u \equiv +\infty$. (Constructively, this interval need not have endpoints, but we overlook that and work classically for the moment.)

The extension of the arithmetic operations to such intervals was defined by Ramon Moore [Moo66]:

$$\begin{array}{lcll}
[d,u] + [e,t] & \equiv & [d+e, u+t] & \\
-[d,u] & \equiv & [-u, -d] & \\
[d,u] \star [e,t] & \equiv & [\mathsf{min}(de, dt, ue, ut),\ \mathsf{max}(de, dt, ue, ut)] & \\
[d,u]^{-1} & \equiv & [u^{-1}, d^{-1}] & \text{if } 0 \notin [d,u], \text{ so } 0 \in D \cup U \\
 & \equiv & [-\infty, +\infty]. & \text{if } 0 \in [d,u]
\end{array}$$

The formula for multiplication is complicated by the need to consider all combinations of signs; we give its (even more complicated) constructive version in Section 10.

Remark 3.5 Moore's ***interval analysis*** has since been used to develop a variety of numerical algorithms. Amongst these, we focus on what it achieves for the problem of ***optimisation***, by which we understand finding the ***maximum value*** of a continuous function defined on a non-empty compact domain, but not necessarily a *location* where the function attains that value. Plainly, *any* value of the function provides a *lower* bound for the maximum, but finding *upper* bounds is problematic using standard numerical methods, especially when the function has "spikes".

For the sake of illustration, we consider an arithmetical function $f : [0,1]^n \to \mathbb{R}$. If this is simply addition or multiplication, Moore's interval operations provide the result (the minimum and maximum values of the function on the domain) directly. For a more complicated arithmetical function, we interpret the operations according to Moore's formulae, and may (if we're lucky) still obtain the minimum and maximum.

In general, however, the result of Moore's interpretation will be an interval that *contains* the required image. In other words, it provides an *upper* bound of the maximum — exactly what standard numerical methods find difficult. Unfortunately, this may be a *vast* over-estimate, especially when the definition of the function is complicated and involves subtraction of similar large numbers, so this doesn't really help very much with spiky functions.

In fact, Moore and his followers have provided various techniques for reducing the sizes of the resulting intervals. One of these simply massages the arithmetic expression to reduce multiple occurrences of variables and sub-expressions, as computing $x - x$ introduces a large error that is easily avoided. However, these techniques are not the point of the present discussion.

Remark 3.6 We regard Moore's definitions as merely *one* way of extending certain continuous functions from $\mathbb{R}$ to $\Sigma^{\mathbb{Q}} \times \Sigma^{\mathbb{Q}}$, as we required above. In fact, that's exactly the point:

(a) *ideally*, we extend $f : \mathbb{R} \to \mathbb{R}$ to the function $F_0 : \Sigma^{\mathbb{Q}} \times \Sigma^{\mathbb{Q}}$ that is defined by

$$F_0[d, u] \;\equiv\; \{fx \mid x \in [d, u]\},$$

(b) but in practice, Moore's operations extend a general arithmetic expression f to *some* continuous endofunction F of $\Sigma^{\mathbb{Q}} \times \Sigma^{\mathbb{Q}}$ such that

$$F[a, a] \;=\; [fa, fa] \quad \text{and} \quad F_0[d, u] \subset F[d, u].$$

The first question that this raises is whether the ideal function F_0, which (a) defines set-theoretically, is *continuous*. Then we can ask whether there is a way of *computing* it, using (b). Remember, nevertheless, that the *advantage* of Moore's interpretation is that it preserves (syntactic) composition of arithmetic expressions, so it is easily performed by a compiler

As we have said, the fundamental problem is the extension of open subsets, which may involve parameters (Remark 3.3). We introduce a new operation I that does this in a *uniform* way, instead of a merely existential property of the extension of functions or open subspaces *one at a time*. This amounts to saying that the open subset can itself be a parameter.

Proposition 3.7 The map $I : \Sigma^{\mathbb{R}} \rightarrowtail \Sigma^{\Sigma^{\mathbb{Q}} \times \Sigma^{\mathbb{Q}}}$ defined by

$$(U \subset \mathbb{R}) \text{ open } \;\mapsto \lambda \delta \upsilon.\, \exists du.\, \delta d \wedge \upsilon u \wedge \big([d, u] \subset U\big)$$

is Scott-continuous, and makes $\Sigma^{\mathbb{R}}$ a retract of $\Sigma^{\Sigma^{\mathbb{Q}} \times \Sigma^{\mathbb{Q}}}$, as it satisfies the equation

$$x : X,\ \phi : \Sigma^{\mathbb{R}} \;\vdash\; I\phi(ix) \;\Leftrightarrow\; \phi x.$$

Proof Heine–Borel theorem, *i.e.* the "finite open sub-cover" definition of compactness for the closed interval $[d, u]$, says exactly that the expression $\big([d, u] \subset U\big)$ is a Scott-continuous predicate in the variable $U : \Sigma^{\mathbb{R}}$. This predicate is also parametric in $d, u \in \mathbb{Q}$, so Definition 2.3 interprets the whole expression for I as a Scott-continuous function of U. This satisfies the equation because, if ϕ classifies $U \subset X$,

$$\phi a \;\equiv\; (a \in U) \;\mapsto\; I\phi(ia) \;\equiv\; \big(\exists du.\, a \in (d, u) \subset [d, u] \subset U\big) \;\Longleftrightarrow\; (a \in U),$$

which expresses *local compactness* of $\mathbb{R}$. ∎

Proposition 3.8 The idempotent $\mathcal{E} \equiv I \cdot \Sigma^{i}$ on $\Sigma^{\Sigma^{\mathbb{Q}} \times \Sigma^{\mathbb{Q}}}$ is given by

$$\mathcal{E}\Phi(\delta, \upsilon) \;\Leftrightarrow\; \exists q_0 < \cdots < q_{2n+1}.\;\; \delta q_1 \wedge \upsilon q_{2n} \;\wedge\; \bigwedge_{k=0}^{n-1} \Phi(\lambda e.\, e < q_{2k},\; \lambda t.\, q_{2k+3} < t).$$

Proof By construction we have

$$\mathcal{E}\Phi(\delta, \upsilon) \;\equiv\; I\big(\lambda x.\, \Phi(ix)\big)(\delta, \upsilon) \;\Leftrightarrow\; \exists du.\, \delta d \wedge \upsilon u \wedge \forall x{:}[d, u].\; \Phi(\delta_x, \upsilon_x) : \Sigma^{\Sigma^{\mathbb{Q}} \times \Sigma^{\mathbb{Q}}}.$$

Now, $\Phi : \Sigma^{\mathbb{Q}} \times \Sigma^{\mathbb{Q}} \to \Sigma$ is a Scott-continuous function, and

$$(D_x, U_x) \;=\; \bigcup_{e<x<t}^{\uparrow} (D_e, U_t)$$

is a directed join in $\Sigma^{\mathbb{Q}} \times \Sigma^{\mathbb{Q}}$, so Φ preserves it, *i.e.*

$$\Phi(ix) \equiv \Phi(\delta_x, \upsilon_x) \Leftrightarrow \exists et.\, (e < x < t) \wedge \Phi(\delta_e, \upsilon_t).$$

Hence the compact interval $[d, u]$ is covered by open intervals (e, t) for each of which $\Phi(\delta_e, \upsilon_t)$ holds. By the Heine–Borel theorem, finitely many ($n \geq 1$) of them suffice, whilst they must overlap by connectedness. So we may adopt a more explicit notation for them:

$$[d, u] \equiv [q_1, q_{2n}] \subset (q_0, q_3) \cup (q_2, q_5) \cup (q_4, q_7) \cup \cdots \cup (q_{2n-2}, q_{2n+1}).$$

Using this, $\forall x{:}[d, u].\ \Phi(\delta_x, \upsilon_x)$ is

$$\exists q_0 < q_1 \equiv d < q_2 < \cdots < q_{2n-1} < q_{2n} \equiv u < q_{2n+1}.\ \bigwedge_{k=0}^{n-1} \Phi(\lambda e.\, e < q_{2k},\ \lambda t.\, q_{2k+3} < t)$$

and then $\mathcal{E}\Phi(\delta, \upsilon)$ is as stated. □

The formula $\mathcal{E}$ will be used to define the real line itself as an object in ASD in Section 7, but it will turn out to be important for many other reasons. In particular, it shows how Moore's interval arithmetic solves the optimisation problem:

Corollary 3.9 Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function and F be *any* continuous extension of f to $\Sigma^{\mathbb{Q}} \times \Sigma^{\mathbb{Q}}$. For example, if f is an arithmetic expression then F may be its interpretation using Moore's operations. Suppose that e and t are strict bounds for the image of $[d, u]$ under f, so

$$\forall x{:}[d, u].\ e < fx < t, \quad \text{or, equivalently,} \quad F_0[d, u] \subset (e, t).$$

Then there is some finite subdivision $d \equiv m_0 < m_1 < \cdots < m_{n-1} < m_n \equiv u$ such that

$$(e, t) \supset \bigcup_{k=0}^{n-1} F[m_k, m_{k+1}].$$

Moreover, we obtain F_0 from F in Remark 3.6 using $\mathcal{E}$ and Remark 3.3. □

Remark 3.10 Notice that we can *write* the formula for $\mathcal{E}$ in Proposition 3.8 with very little assumption about the underlying logic. The role of the classical Heine–Borel theorem was to define a different map, I, and to prove that it is Scott-continuous and satisfies the equations

$$\Sigma^i \cdot I = \mathsf{id}_{\Sigma^{\mathbb{R}}} \quad \text{and} \quad I \cdot \Sigma^i = \mathcal{E}.$$

It is the formula $\mathcal{E}$ — not the classically defined map I or the accompanying proof — that we shall use to construct the real line in ASD. Then the "universal quantifier" that we shall take as the definition of compactness in Definition 4.14 and Section 11 will be based on essentially the same idea as the Corollary. The existential quantifier over the whole line will also be derived from $\mathcal{E}$ in Proposition 8.1.

Remark 3.11 The Corollary justifies the interval-subdivision algorithms that have been developed using interval analysis, but has much more profound consequences for that subject.

Definition 1.1 completely axiomatises $\mathbb{R}$. We intend to give a practical justification of this claim by developing the basic results of analysis from it in future work. But that Definition says that the arithmetic operations and relations and the quantifiers are enough to capture $\mathbb{R}$, whilst we have just indicated how the quantifiers may be computed (*i.e.* eliminated) by extending the arithmetic operations *à la* Moore to intervals. This means that the analysis that we develop using this axiomatisation can be translated into algorithms for real computation in the style of interval analysis.

4 The ASD lambda calculus

This section and the next summarise the symbolic language for ASD in a "user manual" style. The development of the rest of this paper will be conducted entirely within this calculus. We repeat that there is no underlying set theory, and that the monadic and Phoa principles are not negotiable, although there are other formulations of Scott continuity of various strengths besides the one we give here.

Axiom 4.1 The ***types*** of ASD are generated from
(a) base types $\mathbf{1}$, Σ and $\mathbb{N}$, by
(b) products, $X \times Y$,
(c) exponentials, Σ^X, or $X \to \Sigma$ if you prefer, and
(d) Σ-split subspaces (next section).

We do not introduce general exponentials Y^X, infinitary products, type variables or dependent types (at least, not in this version of the calculus). In Section 11 we shall talk about open, closed, compact and overt subspaces of R, maybe with parameters, but we represent them as terms, without requiring new (dependent) types in the underlying type theory.

Axiom 4.2 The ***logical terms*** of type Σ or Σ^U (also known as ***propositions*** and ***predicates*** respectively) are generated from
(a) constants $\top$ and $\bot$;
(b) variables of all types, for which we use increasingly exotic alphabets as the types get more complex;
(c) the lattice connectives $\wedge$ and $\vee$ (but not $\Rightarrow$ or $\neg$);
(d) pairing $\langle\ ,\ \rangle$ and projections π_0, π_1;
(e) λ-abstraction $\lambda x.\, \phi$, where ϕ must itself be logical (*i.e.* of some type of the form Σ^V, and in particular not $\mathbb{N}$ or $\mathbb{R}$), but x may be of any type;
(f) λ-application ϕa, where $a : A$, and $\phi : \Sigma^A$ is logical;
(g) equality $(n =_N m)$, where N is a *discrete* type;
(h) inequality or apartness $(h \neq_H k)$, where H is a *Hausdorff* type;
(i) existential quantification $\exists x{:}X.\, \phi x$, where ϕx must be logical, and the type X must be *overt*;
(j) universal quantification $\forall k{:}K.\, \phi k$, where ϕk must be logical, and the type K must be *compact*.

Discrete, Hausdorff, overt and compact spaces will be defined shortly. We shall see that existential quantification over $\mathbb{N}$ or $\mathbb{R}$ is allowed, but universal quantification isn't. Universal quantification over the *closed* interval $\mathbb{I} \equiv [0, 1]$ is OK; it is justified in Section 11.

So examples of *invalid* logical formulae include $(\lambda n.\, n{+}3)$, $(\lambda x.\, \sqrt[3]{x})$, $(\pi =_{\mathbb{R}} 3.14159)$, $(\forall n.\, \exists pq.\, 2n = p{+}q)$ and $(\forall x.\, x^2 \neq -1)$, but $(\exists npqr.\, p^n + q^n = r^n)$ and $(\forall x{:}|x| \leq 2.\, x^2 < 0)$ are fine. Programs of type `nat` $\to$ `nat` or `real` $\to$ `real` are denoted by *partial* functions, or total functions with different types, such as $\mathbb{N}_\bot$ [D, F].

Binding, renaming, duplication, omission and substitution for variables are as standard in the λ- and predicate calculi. A quantified formula has the same type as its body (ϕ), whilst λ-abstraction and application modify types in the usual way. Alternating quantifiers are allowed to any depth — so long as their ranges are overt or compact spaces.

Axiom 4.3 The ***numerical terms*** of type $\mathbb{N}$ are generated from
(a) the constant 0;
(b) variables;
(c) successor;
(d) definition by description: `the` $n.\, \phi n$ (Axiom 4.8).
(e) We also allow *primitive recursion* over $\mathbb{N}$ at *any* type X.

Remark 4.4 We shall take for granted the usual arithmetic operations on $\mathbb{N}$, $\mathbb{Z}$ and $\mathbb{Q}$, and the operations of *naïve* set theory over these types, including, for example, both quantifiers over any list or (Kuratowski-) finite set of rationals, and existential quantification over the type of lists. (See [E] for more about lists in ASD.) These operations have well known definitions using primitive recursion. The infinite objects that are generated in this way are ***countable*** in the sense of being isomorphic to $\mathbb{N}$ in the calculus; since isomorphisms are homeomorphisms, these objects inherit $=$, $\neq$ and $\exists$ from $\mathbb{N}$.

Definition 4.5 ***Judgements*** in the calculus are of the four forms

$$\vdash\ X\ \mathsf{type}, \qquad \Gamma\ \vdash\ a : X, \qquad \Gamma\ \vdash\ a = b : X \quad \text{and} \quad \Gamma\ \vdash\ \alpha \leq \beta : \Sigma^X.$$

asserting well-formedness of types and typed terms, and equality or implication of terms. We refer to $a = b$ and $a \leq b$ on the right of the $\vdash$ as ***statements***. On the left is the ***context*** Γ, which consists of assignments of types to variables, and possibly also equational hypotheses [E, §2]. The form $\Gamma\ \vdash\ \alpha \leq \beta : \Sigma^X$ is syntactic sugar.

Axiom 4.6 The predicates and terms satisfy certain ***equational axioms***, including

(a) those for a distributive lattice; in particular, for $\Gamma \vdash \phi, \psi : \Sigma^X$ we write

$$\phi\ \leq\ \psi \quad \text{to mean} \quad \phi \wedge \psi\ =\ \phi, \quad \text{or equivalently} \quad \phi \vee \psi\ =\ \psi,$$

although for $\Gamma \vdash \sigma, \tau : \Sigma$ we use $\sigma \Rightarrow \tau$ and $\sigma \Leftrightarrow \tau$ instead of $\sigma \leq \tau$ and $\sigma = \tau$, since we shall also need the symbols $\leq$ and $=$ for their more usual "numerical" meanings;

(b) the β- and η-rules for λ-abstraction/application and pairing/projection;

(c) the β- and η-rules for primitive recursion over $\mathbb{N}$; and

(d) others that we describe in a little more detail in the rest of this section.

Remark 4.7 Predicates on their own denote open subspaces and continuous functions, but their expressive power is very weak. We introduce nested implications into the calculus by making their hierarchy explicit, in the form of statements and judgements.

(a) Observe, first, that $\leq$, $=$, $\Rightarrow$ and $\Leftrightarrow$ link *predicates* to form *statements*, not new predicates. In other words, Σ is a lattice and not a Heyting algebra.

(b) There are equality *statements* $a = b : X$ for *any* type, but equality *predicates* $(n =_N m) : \Sigma$ only for *discrete* types. In particular, for $a, b : R$, there are *predicates* $a \neq b$ and $a < b$, but $a = b$ and $a \leq b$ are *statements*, since R is Hausdorff but not discrete.

(c) For any predicate α, we sometimes write α or $\neg\alpha$ for the *statements* $\alpha \Leftrightarrow \top$ or $\alpha \Leftrightarrow \bot$.

(d) Nested equality and implication are not allowed in statements: we have to use a *judgement* of the form $\alpha_2 \Rightarrow \beta_1 \vdash \gamma_1 \Rightarrow \delta_0$ instead. If we require another level of nesting, we must use a proof rule:

$$\frac{\alpha_3 \Rightarrow \beta_2 \vdash \gamma_2 \Rightarrow \delta_1}{\epsilon_2 \Rightarrow \zeta_1 \vdash \eta_1 \Rightarrow \theta_0}$$

See Exercise 6.15 for examples. This is as far as we can go. Plainly there's something artificial about this limit, which will be rectified in an extended calculus in future work.

(e) Equality and implication statements for function-types Σ^X provide a way of stating quantification over any type X — as a statement — it's a predicate only when X is compact.

Our convention is that λ-application binds most tightly, followed by the propositional relations, then $\wedge$, then $\vee$, then λ, $\exists$ and $\forall$, then $\Rightarrow$, $\Leftrightarrow$, $\leq$, $=$ and finally $\vdash$. This reflects the hierarchy of propositions, predicates, statements, judgements and rules. We often bracket propositional equality for emphasis and clarity.

Axiom 4.8 A predicate $\Gamma,\ n : \mathbb{N}\ \vdash\ \phi n : \Sigma$ is called a ***description*** if it is "uniquely satisfiable" in the sense that

$$\Gamma\ \vdash\ (\exists n.\, \phi n)\ \Leftrightarrow\ \top \quad \text{and} \quad \Gamma,\ n, m : \mathbb{N}\ \vdash\ (\phi n \wedge \phi m)\ \Rightarrow\ (n =_{\mathbb{N}} m)$$

are provable. Then we may introduce $\Gamma \vdash \mathsf{the}\, n.\, \phi n : \mathbb{N}$, satisfying

$$\Gamma,\ m : \mathbb{N} \vdash \phi m \Leftrightarrow (m =_{\mathbb{N}} \mathsf{the}\, n.\, \phi n).$$

Since the notion of description characterises *singleton* predicates, $\phi = \{n\} \equiv (\lambda m.\, n = m)$, it is only meaningful for (overt) *discrete* objects, not for $\mathbb{R}$.

In the more abstract parts of ASD, in particular [A, B], every object is required to be *sober* in a sense that is defined in terms of an equaliser of powers of Σ, and an operator focus is added to the λ-calculus. This is actually part of the monadic idea in Section 5. In the presence of the other structure, these things are equivalent to definition of natural numbers by description [A, §§9–10], and general recursion is also definable [D].

Axiom 4.9 In addition to the equations for a distributive lattice, Σ satisfies the ***Phoa principle***,

$$F : \Sigma^\Sigma,\ \sigma : \Sigma \vdash F\sigma \Leftrightarrow F\bot \vee \sigma \wedge F\top : \Sigma.$$

The naïve interpretation of this is that "Σ has only two elements", $\bot$ and $\top$, which denote termination ("yes") and non-termination ("wait") respectively, and that any function $\Sigma \to \Sigma$ is specified by its effect on these two possible inputs. It cannot interchange them, as that would solve the Halting Problem, so $F\bot \Rightarrow F\top$, and more generally all maps $\Sigma^Y \to \Sigma^X$ are monotone with respect to the lattice order. Note that Σ is not a "Boolean" type, as there is no *definite* "no".

Remark 4.10 However, the Phoa principle also captures the "extensional" aspects of topology (Remark 2.6), saying that

open subspaces of X, closed subspaces of X and continuous functions $X \to \Sigma$

are in bijection. In particular, if the inverse images of $\top$ under $\phi, \psi : X \rightrightarrows \Sigma$ coincide as (open) subspaces of X then $\phi = \psi$ as logical terms. Likewise the inverse images of $\bot$ [C].

We say that ϕ ***classifies*** the corresponding open subspace, and ***co-classifies*** the closed one. In symbols, for $a : X$,

$$a \in U \text{ (open) iff } \phi a \Leftrightarrow \top \quad \text{and} \quad a \in C \text{ (closed) iff } \phi a \Leftrightarrow \bot.$$

In fact, such inverse images do not exist as types generated from $\mathbb{N}$ and Σ by $\times$ and $\Sigma^{(-)}$: they are examples of Σ-split subspaces (Section 11).

A ***clopen***, ***complemented*** or ***decidable*** subspace is one that is both open and closed, so it and its complement are the inverse images of $0, 1 \in \mathbf{2}$ under some (unique) map $X \to \mathbf{2}$ [C, Proposition 9.6].

Lemma 4.11 The Phoa principle justifies rules for negation that resemble those of Gerhard Gentzen's classical sequent calculus [Gen35]:

$$\frac{\Gamma,\ \sigma \Leftrightarrow \top \vdash \alpha \Rightarrow \beta}{\Gamma \vdash \sigma \wedge \alpha \Rightarrow \beta} \qquad \frac{\Gamma,\ \sigma \Leftrightarrow \bot \vdash \beta \Rightarrow \alpha}{\Gamma \vdash \beta \Rightarrow \sigma \vee \alpha}$$

Proof The rule on the left says that the intersection of the *open* subspaces of Γ classified by σ and α is contained in that classified by β. The rule on the right says exactly the same thing, but for the intersection of *closed* subspaces. □

Subspaces of $X^{\mathbf{2}}$, $X^{\mathbf{3}}$ *etc.* are often called binary, ternary, ... ***relations***. In particular, open binary relations are the same thing as predicates with two variables.

Definition 4.12 If the diagonal subspace, $X \subset X \times X$, is open or closed then we call X ***discrete*** or ***Hausdorff*** respectively. Type-theoretically, such spaces are those in which we may internalise equality-*statements* as *predicates*:

$$\frac{\Gamma \vdash n = m : N}{\Gamma \vdash (n =_N m) \Leftrightarrow \top : \Sigma} \qquad \frac{\Gamma \vdash h = k : H}{\Gamma \vdash (h \neq_H k) \Leftrightarrow \bot : \Sigma}$$

Lemma 4.13 Equality has the usual properties of substitution, reflexivity, symmetry and transitivity, whilst inequality or apartness obeys their lattice duals:

$$\begin{array}{rcl} \phi m \wedge (n = m) & \Rightarrow & \phi n \\ (n = n) & \Leftrightarrow & \top \\ (n = m) & \Leftrightarrow & (m = n) \\ (n = m) \wedge (m = k) & \Rightarrow & (n = k) \end{array} \qquad \begin{array}{rcl} \phi h \vee (h \neq k) & \Leftarrow & \phi k \\ (h \neq h) & \Leftrightarrow & \bot \\ (h \neq k) & \Leftrightarrow & (k \neq h) \\ (h \neq k) \vee (k \neq \ell) & \Leftarrow & (h \neq \ell) \end{array}$$

In an overt discrete space, or a compact Hausdorff one, we have the converse of the first of the four rules:

$$\exists m.\, \phi m \wedge (n = m) \;\Leftrightarrow\; \phi n \qquad \forall h.\, \phi h \vee (h \neq k) \;\Leftrightarrow\; \phi k.$$

When *both* $=_X$ and $\neq_X$ are defined, as they are for $\mathbb{N}$, they are complementary:

$$(n =_X m) \vee (n \neq_X m) \;\Leftrightarrow\; \top \quad \text{and} \quad (n =_X m) \wedge (n \neq_X m) \;\Leftrightarrow\; \bot.$$

In this case X is said to have ***decidable equality***. □

Definition 4.14 A space X that admits existential or universal quantification is called ***overt*** or ***compact*** respectively. By these quantifiers we mean the type-theoretic rules

$$\frac{\Gamma,\ x : X \ \vdash\ \phi x \ \Rightarrow\ \sigma : \Sigma^U}{\Gamma \ \vdash\ \exists x.\, \phi x \ \Rightarrow\ \sigma : \Sigma^U} \qquad \frac{\Gamma,\ x : X \ \vdash\ \sigma \ \Rightarrow\ \phi x : \Sigma^U}{\Gamma \ \vdash\ \sigma \ \Rightarrow\ \forall x.\, \phi x : \Sigma^U}$$

Remark 4.15 So long as the types of the variables really are overt or compact, and we observe the caveats in Remarks 2.5 and 12.4, we may reason with the quantifiers in the usual ways:

(a) If we find a particular $\Gamma \vdash a : X$ that satisfies $\Gamma \vdash \phi a \Leftrightarrow \top$, then we may of course assert $\Gamma \vdash \exists x.\, \phi x \Leftrightarrow \top$. This simple step tends to pass unnoticed in the middle of an argument, often in the form $\phi a \Rightarrow \exists x.\, \phi x$.

(b) Similarly, if the judgement $\Gamma \vdash \forall x.\, \phi x \Leftrightarrow \top$ has been proved, and we have a particular value $\Gamma \vdash a : X$, then we may deduce $\Gamma \vdash \phi a \Leftrightarrow \top$. Again, we often write $\phi a \Leftarrow \forall x.\, \phi x$.

(c) The familiar mathematical idiom "there exists", in which $\exists x.\, \phi x$ is asserted and then x is temporarily used in the subsequent argument, is valid, as [Tay99, §1.6] explains.

(d) The λ-calculus formulation automatically allows substitution under the quantifiers, whereas in categorical logic this property must be stated separately, and is known as the Beck–Chevalley condition [Tay99, Chapter IX].

Exercise 4.16 Use the Phoa principle to prove the ***Frobenius*** and ***modal laws***

$$\begin{array}{rcl} \exists x.\, \sigma \wedge \phi x & \Leftrightarrow & \sigma \wedge \exists x.\, \phi x \\ \forall x.\, \sigma \vee \phi x & \Leftrightarrow & \sigma \vee \forall x.\, \phi x \end{array} \qquad \begin{array}{rcl} (\forall x.\, \phi x) \wedge (\exists x.\, \psi x) & \Rightarrow & \exists x.\, (\phi x \wedge \psi x) \\ (\forall x.\, \phi x) \vee (\exists x.\, \psi x) & \Leftarrow & \forall x.\, (\phi x \vee \psi x) \end{array}$$

where the type of x is both overt and compact. The Frobenius law for $\forall$ is another feature that ASD has in common with classical but not intuitionistic logic; it was nevertheless identified in intuitionistic locale theory by Japie Vermeulen [Ver94]. □

Lemma 4.17 Any topology Σ^X has joins indexed by overt objects and meets indexed by compact ones:

$$\bigvee_N \equiv \exists_N^X : (\Sigma^X)^N \cong (\Sigma^N)^X \to \Sigma^X \quad \text{and} \quad \bigwedge_K \equiv \forall_K^X : (\Sigma^X)^K \cong (\Sigma^K)^X \to \Sigma^X.$$

Binary meets distribute over joins by the Frobenius law, and "substitution under the quantifier" means that all inverse image maps $\Sigma^f : \Sigma^Y \to \Sigma^X$, where

$$\Sigma^f \psi \;\equiv\; \psi \cdot f \;\equiv\; \lambda x.\, \psi(fx) \quad \text{or} \quad \Sigma^f V \;\equiv\; f^* V \;\equiv\; f^{-1} V \;\equiv\; \{x \mid \psi x\},$$

preserve all joins indexed by overt objects, and meets indexed by compact ones [C]. □

Remark 4.18 We often want the quantifiers or meets and joins to range over dependent types, even though we haven't provided these in the calculus.

The most pressing case of this is the join or existential quantifier indexed by an open subspace $M \subset N$ of an overt space. This subspace is classified by a predicate $\alpha : \Sigma^N$, which we shall write as $\Gamma,\ n : N \ \vdash\ \alpha_n : \Sigma$. The M-indexed family $\phi^m : \Sigma^X$ of which we want to form the join may always be considered to be the restriction to M of an N-indexed family, so we have

$$\bigvee_{m:M} \phi^m \ \equiv\ \exists n{:}N.\ \alpha_n \wedge \phi^n : \Sigma^X.$$

The sub- and super-script notation indicates co- and contra-variance with respect to an imposed order relation on N.

We shall also want to define both quantifiers over the closed interval $[d, u]$, where d and u are variable, but we shall cross this bridge when we come to it, in Section 11.

Definition 4.19 Such a pair of families (α_n, ϕ^n) is called a ***directed diagram***, and the corresponding

$$\bigvee^{\uparrow}_{n:\alpha_n} \phi^n \quad\equiv\quad \exists n.\ \alpha_n \wedge \phi^n$$

is called a ***directed join***, if (a) $(\exists n.\ \alpha_n) \Leftrightarrow \top$, and (b) $\alpha_{n@m} \Leftrightarrow \alpha_n \wedge \alpha_m$ and $\phi^{n@m} \geq \phi^n \vee \phi^m$ for some binary operation $@ : N \times N \to N$.

In this, $\alpha_n \wedge \alpha_m$ means that both ϕ^n and ϕ^m contribute to the join, so for directedness in the informal sense, we require some $\phi^{n@m}$ to be above them both (contravariance), and also to count towards the join, for which $\alpha_{n@m}$ must be true (covariance). Hence the imposed order relation on N is that for which $(N, @)$ is a essentially a meet semilattice.

Axiom 4.20 Any $\Gamma \vdash F : \Sigma^{\Sigma^X}$ is ***Scott-continuous***, *i.e.* it preserves directed joins in the sense that

$$\Gamma \vdash\ F(\exists n.\ \alpha_n \wedge \phi^n) \ \Leftrightarrow\ \exists n.\ \alpha_n \wedge F\phi^n.$$

Notice that F is attached to ϕ^n and not to α_n, since the join being considered is really that over the subset $M \equiv \{n \mid \alpha_n\} \subset N$. The principal use of this axiom in this paper is Proposition 7.9, where the ambient overt object N is $\mathbb{Q}$, and its imposed order is either the arithmetic one or its reverse, so @ is either max or min. Scott continuity is also what connects our definition of compactness with the traditional "finite open sub-cover" one (Remark 11.12), whilst in denotational semantics it gives a meaning to recursive programs.

Axiom 4.21 $\mathbb{N}$ is (discrete, Hausdorff and) overt but not compact, so each Σ^X has and each Σ^f preserves $\mathbb{N}$-indexed joins but not $\mathbb{N}$-indexed meets. This and Scott continuity break the de Morgan-style lattice duality that all of the preceding rules enjoyed.

Examples 4.22 We conclude with examples of the four main properties.

	overt	discrete	compact	Hausdorff
$\mathbb{N} \times \Sigma$	✓	×	×	×
$\mathbb{R}$, $\mathbb{R}^{\mathbf{n}}$	✓	×	×	✓
Σ	✓	×	✓	×
$\mathbb{I}$, $\mathbf{2}^{\mathbb{N}}$	✓	×	✓	✓
free combinatory algebra	✓	✓	×	×
$\mathbb{N}$, $\mathbb{Q}$	✓	✓	×	✓
Kuratowski finite	✓	✓	✓	×
finite ($\mathbf{n}$)	✓	✓	✓	✓
the set of codes of non-terminating programs	×	✓	×	✓

The axioms only provide the quantifiers $\forall_\emptyset \equiv \top$, $\exists_\emptyset \equiv \bot$, $\forall_{\mathbf{2}} \equiv \wedge$, $\exists_{\mathbf{2}} \equiv \vee$ and $\exists_{\mathbb{N}}$ directly — it is the business of this paper to *construct* $\exists_{\mathbb{R}}$ $\exists_{\mathbb{I}}$ and $\forall_{\mathbb{I}}$, in Sections 8 and 11. To put this the other way round, assuming that the future extension of the calculus will allow arbitrary nesting of $\Rightarrow$ and $\forall$ in statements, these constructions will justify ***quantifier elimination*** in the corresponding cases.

5 The monadic principle

Remark 5.1 In approaches to topology based on sets of points — including those in which the word "set" is replaced by (Martin–Löf) *type* or *object* (of a topos), or "functions" are accompanied by *programs* — new objects are constructed by specifying a predicate on a suitable ambient object, and then carving out the desired subobject using the subset-forming tools provided by the foundations. For example, the real line may be constructed as the *set* (type, object) of those pairs (D, U) or (δ, υ) that satisfy the properties required of a Dedekind cut, which we shall formalise in the next section.

Such approaches fall on the mercy of the underlying logic of sets of points to prove important theorems of mathematics, such as compactness of the real closed interval, and specifically Proposition 3.7. As we shall see in Section 12, in many interesting logical systems, especially those that are designed to capture recursion, the logic may not oblige us with the proof of such theorems, and indeed may falsify them.

Abstract Stone Duality sets itself apart from these approaches by forming subspaces in a way that takes account of their intended topology.

Remark 5.2 In this respect ASD inherits the philosophy of locale theory, which abandons the points altogether, developing topology entirely in terms of lattices of open subsets. Famously, this avoids many of the uses of the axiom of choice that plague point-set topology, notably Tychonov's theorem about products of compact spaces [Joh82, Theorem III 1.7]. But it also secures compactness of the real closed interval, even in sheaf toposes where this fails when the real line is defined in terms of its points [FH79] [Joh82, IV 1.3]. On the other hand, the lattice of open subspaces is still a set (type, object) with imposed structure, whereas the foundations of ASD are recursive.

Remark 5.3 Proposition 3.7 gave the leading example of the way in which we can define a subspace along with its topology. In order to abstract the structure of which it is an example, we need a category with (finite products and) powers Σ^X of a fixed object Σ. Such a category is captured symbolically by the types that we specified in Axiom 4.1, together with the restricted λ-calculus that is associated with them.

Beware, however, that not all subspaces (equalisers, Definition 7.2) have the relevant property, for example $i : \mathbb{N}^{\mathbb{N}} \rightarrowtail \Sigma^{\mathbb{N}\times\mathbb{N}}$ can be expressed as an equaliser of topological spaces or locales, but it doesn't have a continuous Σ-splitting I. Retracts and open and closed subspaces do, quite simply (Section 11). As we shall see in Lemma 11.9, there may be many different Σ-splittings for the same inclusion.

The subspace calculus is therefore not as easy to use as set-theoretic comprehension — the rule has been to introduce only one subspace per paper. So far, it only gives a robust account of locally compact spaces, but work has begun on extending it to and beyond general locales [H].

Definition 5.4 An inclusion $i : X \rightarrowtail Y$ is called a Σ***-split subspace*** if there is another map $I : \Sigma^X \rightarrowtail \Sigma^Y$ such that

$$x : X,\ \phi : \Sigma^X \ \vdash\ I\phi(ix) \ \Leftrightarrow\ \phi x,$$

thereby making Σ^X a retract of Σ^Y. Hence the idempotent $E \equiv \Sigma^i \cdot I : \Sigma^Y \to \Sigma^Y$ determines Σ^X, and therefore X itself as the equaliser of the maps $Y \rightrightarrows \Sigma^{\Sigma^Y}$ given by $y \mapsto \lambda\psi.\,\psi y$ and $\lambda\psi.\,E\psi y$. (From Axiom 4.8 it can be shown that the equaliser is indeed the original X, and not some other object X' with $\Sigma^{X'} \cong \Sigma^X$.)

Not every idempotent E on Σ^Y can arise in this way from a Σ-split subspace, because the surjective part Σ^i of its splitting must be a homomorphism for the topological structure, in particular for $\wedge$ and $\vee$. Since any order-preserving surjective function between lattices preserves $\top$ and $\bot$ anyway, the crucial condition is this:

Lemma 5.5 $E \equiv \Sigma^i \cdot I : \Sigma^Y \to \Sigma^Y$ satisfies the equations

$$\phi, \psi : \Sigma^Y \ \vdash \ E(\phi \wedge \psi) \ = \ E(E\phi \wedge E\psi) \quad \text{and} \quad E(\phi \vee \psi) \ = \ E(E\phi \vee E\psi). \qquad \square$$

We shall explain why these equations are sufficient later, but we are now ready to give the rules of this part of the calculus, at least in "user manual" style. It *formally adjoins* Σ-split subspaces to a *category*, just as number-theorists formally adjoin roots of particular polynomials to a *field*, or set theorists construct "equiconsistent" *models* that have altered properties.

Definition 5.6 A term $y : Y,\ \psi : \Sigma^Y \vdash E\psi y : \Sigma$ that satisfies the equations in the Lemma is called a ***nucleus***. In order to avoid developing a theory of dependent types, we do not allow E to have parameters (besides y and ψ).

The word "nucleus" was appropriated from locale theory, since both kinds of nuclei play the same role, namely to define subspaces, but the definitions are different. A localic nucleus, usually called j, must satisfy $\mathsf{id} \leq j = j \cdot j$ but needn't be Scott-continuous. Nuclei in ASD, on the other hand, are continuous but needn't be order-related to id. So the common ground is when $E \geq \mathsf{id}$ and j is continuous.

Next we identify the members of the subspace of Y that E defines.

Definition 5.7 A term $\Gamma \vdash a : Y$ (which may now involve parameters) of the larger space Y is called ***admissible*** with respect to the nucleus E if

$$\Gamma,\ \psi : \Sigma^Y \ \vdash \ \psi a \ \Leftrightarrow \ E\psi a.$$

In this case we may regard a as a term of type X, although we write $\mathsf{admit}_{Y,E}\, a$ for it in the formal calculus, in order to disambiguate its type.

The idea behind this definition is this: the operator E "normalises" open subspaces of Y by restricting them to X using Σ^i, and then re-expanding them using I. Therefore $a : Y$ belongs to X iff its membership of *any* open subspace ψ of Y is unaffected by this normalisation.

As in type theory, we call the following rules after the connective that they introduce or eliminate, in this case $\{\ |\ \}$ and $\Sigma^{\{|\}}$.

Axiom 5.8 The $\{\}$-rules of the ***monadic λ-calculus*** define the subspace itself.

$$\frac{Y\ \mathsf{type} \qquad y : Y,\ \psi : \Sigma^Y \ \vdash \ E\psi y : \Sigma \qquad E \text{ is a nucleus}}{\{Y \mid E\}\ \mathsf{type}} \qquad \{\}F$$

$$\frac{\Gamma \ \vdash \ a : Y \qquad \Gamma,\ \psi : \Sigma^Y \ \vdash \ \psi a \ \Leftrightarrow \ E\psi a}{\Gamma \ \vdash \ \mathsf{admit}_{Y,E}\, a : \{Y \mid E\}} \qquad \{\}I$$

$$x : \{Y \mid E\} \ \vdash \ i_{Y,E} x : Y \qquad \{\}E_0$$

$$x : \{Y \mid E\},\ \psi : \Sigma^Y \ \vdash \ \psi(i_{Y,E} x) \ \Leftrightarrow \ E\psi(i_{Y,E} x) \qquad \{\}E_1$$

$$\frac{\Gamma \ \vdash \ a : Y \qquad \Gamma,\ \psi : \Sigma^Y \ \vdash \ \psi a \ \Leftrightarrow \ E\psi a}{\Gamma \ \vdash \ a \ = \ i_{Y,E}(\mathsf{admit}_{Y,E}\, a) : Y} \qquad \{\}\beta$$

$$x : \{Y \mid E\} \ \vdash \ x \ = \ \mathsf{admit}_{Y,E}(i_{Y,E} x) : \{Y \mid E\}. \qquad \{\}\eta$$

These rules say that $\{Y \mid E\}$ is the equaliser mentioned in Definition 5.4, namely

$$\{Y \mid E\} \xrightarrow{\ i\ } Y \underset{y \mapsto \lambda\psi.\, E\psi y}{\overset{y \mapsto \lambda\psi.\, \psi y}{\rightrightarrows}} \Sigma^{\Sigma^Y}$$

Axiom 5.9 The following $\Sigma^{\{\}}$-rules say that $\{Y \mid E\}$ has the subspace topology, where $I_{Y,E}$ expands open subsets of the subspace to the whole space.

$$\phi : \Sigma^{\{Y|E\}} \ \vdash\ I_{Y,E}\phi : \Sigma^Y. \qquad \Sigma^{\{\}}E$$

The β-rule says that the composite $\Sigma^Y \twoheadrightarrow \Sigma^{\{Y|E\}} \rightarrowtail \Sigma^Y$ is E:

$$y : Y,\ \psi : \Sigma^Y \ \vdash\ I_{Y,E}\big(\lambda x{:}\{Y \mid E\}.\ \psi(i_{Y,E}x)\big)y \ \Leftrightarrow\ E\psi y. \qquad \Sigma^{\{\}}\beta$$

Notice that this is the only rule that introduces E into terms. The η-rule says that the other composite $\Sigma^{\{Y|E\}} \rightarrowtail \Sigma^Y \twoheadrightarrow \Sigma^{\{Y|E\}}$ is the identity:

$$\phi : \Sigma^{\{Y|E\}},\ x : \{Y \mid E\} \ \vdash\ \phi x \ \Leftrightarrow\ I_{Y,E}\phi(i_{Y,E}x). \qquad \Sigma^{\{\}}\eta$$

This equation was that in Definition 5.4 for a Σ-split subspace.

Remark 5.10 For another structure to admit a valid interpretation, not only must the equaliser displayed in Axiom 5.8 exist, but there must also be a map I in the structure that satisfies the equations $\Sigma^i \cdot I = \mathsf{id}_{\Sigma^X}$ and $I \cdot \Sigma^i = E$, *cf.* Remark 3.10.

Theorem 5.11 Σ-split subspaces interact with the underlying structure of products and exponentials as follows, and also provide stable disjoint unions [B].

$$\begin{array}{lcl}
X & \cong & \{\Sigma^{\Sigma^X} \mid \lambda\mathcal{F}F.\, F\big(\lambda x.\, \mathcal{F}(\lambda\phi.\, \phi x)\big)\} \\
\{X \mid E_0\} \times \{Y \mid E_1\} & \cong & \{X \times Y \mid E_1^X \cdot E_0^Y\} \\
\Sigma^{\{X|E\}} & \cong & \{\Sigma^X \mid \Sigma^E\} \\
\{\{X \mid E_1\} \mid E_2\} & \cong & \{X \mid E_2\} \\
\{X \mid E_0\} + \{Y \mid E_1\} & \cong & \{\Sigma^{\Sigma^X \times \Sigma^Y} \mid \mathsf{E}\} \\
\text{where} \quad \mathsf{E}\mathcal{H}H & \equiv & H\big\langle \lambda x.\, \mathcal{H}(\lambda\phi\psi.\, E_0\phi x), \lambda y.\, \mathcal{H}(\lambda\phi\psi.\, E_1\psi y)\big\rangle.
\end{array}$$

The (stable, effective, overt, discrete) quotient of an overt discrete object by an open equivalence relation is constructed in [C], whilst free monoids and semilattices are obtained in [E] using a fixed point construction for the required nucleus. □

Remark 5.12 If this paper is the first one about ASD that you have seen, you may reasonably imagine that we have cooked up these rules from the observations in Section 3, based on little more than the properties of the real line that first year undergraduates learn. Can this theory of topology even be generalised to $\mathbb{R}^{\mathbf{n}}$? Is it of any use to prove theorems in mathematics other than those that we have mentioned?

Actually, we have told the story so far in reverse historical order. The calculus above was derived from a categorical intuition, and when it was formulated [B], it was a solution in search of a problem. There were no clear ideas of how the real line might be constructed, and certainly no inkling that it might behave in ASD in a radically different way from the established theory of Recursive Analysis (Section 12). That it turned out to do so is a tribute to category theory as a source of mathematical intuitions.

Remark 5.13 By *Stone Duality* we understand a wide collection of mathematical phenomena, surveyed in [Joh82], of which the leading examples are coordinate or algebraic geometry, and

Marshall Stone's representation of any Boolean algebra as the lattice of clopen subspaces of a totally disconnected compact Hausdorff topological space (1936). A key idea is Stone's maxim that "one must always topologize".

The original idea (1993) of *abstract* Stone duality was that such phenomena might be captured by requiring the adjunction $\Sigma^{(-)} \dashv \Sigma^{(-)}$ (which automatically exists whenever powers Σ^X do) to be *monadic*. Based on a theorem of Adolf Lindenbaum and Alfred Tarski (1935), Robert Paré had shown that the contravariant powerset functor in any elementary topos has this property, thereby deriving colimits from limits [Par71]. It also strengthens the notion of repleteness that had arisen in synthetic domain theory [Hyl91, Tay91].

The point is that monads offer a way of formulating (some kinds of) infinitary algebraic theories over *any* category, not just over the category of sets. In order to remove the arbitrary joins from topology, whilst retaining the computable ones, the idea was to express the category of algebras of abstract open subspaces as an algebraic theory over the category $\mathcal{S}$ of spaces, *i.e.* topologising the topology, according to Stone's maxim. Since algebras are also dual to spaces, $\mathcal{S}^{\mathsf{op}}$ is algebraic over $\mathcal{S}$. Besides any elementary topos, the categories of locally compact sober spaces or locales are other examples of this situation.

However, little progress was made on this until another idea came along (1997), namely the Euclidean principle ($F\sigma \wedge \sigma \Leftrightarrow F\top \wedge \sigma$), which is the part of the Phoa principle (Axiom 4.9) that applies to set theory rather than topology. From this the theory blossomed, and in particular the formulations of discrete, Hausdorff and compact spaces were introduced in [C]. What had been called *open* locales [JT84] became much easier to understand as the dual of compact spaces, but it was also clear that they needed a new name (overt).

The original idea of the monadic adjunction was then developed in [A, B], relating it to the work of Hayo Thielecke on the continuation-passing interpretation of computational effects (goto, *etc.*) [Thi97]. The category is constructed both by formally adding such subspaces, and as the opposite of the category of Eilenberg–Moore algebras for the monad (that's *John* Moore and Samuel Eilenberg, 1965). Finally, based on experience of the translation of type theories into category theory [Tay99], the monadic λ-calculus above was devised to be equivalent to the categorical construction. This involved a normalisation theorem that reduces arbitrarily mixed subspaces and exponentials to a single subspace of a complicated exponential.

The notions of sobriety and Σ-split subspaces in ASD come from the conditions in the theorem of Jon Beck (1966) that characterise monadic adjunctions [Tay99, Theorem 7.5.9]. Sobriety says that $\Sigma^{(-)}$ reflects invertibility, nuclei are essentially the same as $\Sigma^{(-)}$-contractible equalisers, Axiom 5.8 says that such equalisers exist and Axiom 5.9 that $\Sigma^{(-)}$ takes them to coequalisers. These conditions are given as equations between higher order λ-terms in [A, B].

In the presence of the other topological structure in the previous section, in particular Scott continuity, these equations are equivalent to much simpler ones involving finitary $\wedge$ and $\vee$. In this form, the defining equations for a nucleus are those in Definition 5.6 [G].

Any definable object X of ASD can be expressed as a Σ-split subspace of $i : X \rightarrowtail \Sigma^{\mathbb{N}}$, where i and I correspond directly to a basis of open subspaces and related compact subspaces. In this way, locally compact spaces and computably continuous functions between them can be understood directly in ASD.

Some of the arguments in the style of the λ-calculus in [A, C, G–] about discrete, Hausdorff, compact and overt objects were subsequently popularised by Martín Escardó [Esc04]. However, he left out the monadic principle. As we shall see, this is crucial to the compactness of the closed interval.

6 Dedekind cuts

Richard Dedekind's construction of the real line [Ded72] is very familiar, but we need to be sure that it can be carried out in our very weak logic. We begin with what we require of the rationals.

Definition 6.1 A ***dense linear order without endpoints*** is an overt Hausdorff object Q with an open binary relation $<$ that is

(a) transitive and interpolative (dense): $p, r : Q \ \vdash \ (p < r) \ \Leftrightarrow \ (\exists q.\, p < q < r)$
(b) extrapolative (without endpoints): $q : Q \ \vdash \ (\exists p.\, p < q) \ \Leftrightarrow \ \top \ \Leftrightarrow \ (\exists r.\, q < r)$
(c) linear (trichotomous): $p, q : Q \ \vdash \ (p \neq q) \ \Leftrightarrow \ (p < q) \vee (q < p).$

The $\exists q$ in the interpolation axiom does not necessarily mean the midpoint $\frac{p+r}{2}$. It might instead mean the "simplest" interpolant, in some sense, as in John Conway's number system [Con76]. In practical computation, the best choice of q may be determined by considerations elsewhere in the problem, *cf.* unification in Prolog (Remark 2.4).

Lemma 6.2 $p, q, r : Q \ \vdash \ (p < q) \ \wedge \ (q < r) \ \Rightarrow \ (p < r) \ \Rightarrow \ (p < q) \ \vee \ (q < r).$ □

Lemma 6.3 If Q is discrete then $=$ and $<$ are decidable, with $(p \not< q) \equiv (p = q) \vee (q < p)$. This is the case when Q is countable ($Q \cong \mathbb{N}$, Remark 4.4), in which case the usual order on $\mathbb{N}$ induces another (well founded) order $\prec$ on Q that we call ***simplicity***. □

Examples 6.4 Such countable Q may consist of
(a) all fractions n/m with $m \neq 0$, where "simpler" fractions have smaller denominators;
(b) dyadic or decimal fractions $n/2^m$ and $n/10^m$;
(c) continued fractions; or
(d) roots of polynomials with integer coefficients, where the notion of simplicity is given by the degree and coefficients of the polynomials;

where in all cases $<$ is the arithmetical order. For the sake of motivation we shall refer to the elements of *any* such countable Q as ***rationals***, but for a particular target computation any of the above structures may be chosen. We keep the dyadic rationals in mind for practical reasons, but they have no preferred role in the definition.

Proposition 6.5 Any two countable dense linear orders without endpoints are order-isomorphic.

Proof The isomorphism $Q_1 \cong Q_2$ is built up from finite lists of pairs by course-of-values recursion and definition by description (Axiom 4.8). Given such a list, we may add another pair that consists of the simplest absentee from Q_1, together with its simplest order-match from Q_2. At the next stage we start with an absentee from Q_2. □

Remark 6.6 Of course, this well known model-theoretic result misses the point of real analysis. When we apply it to different notions of "rationals" it destroys the arithmetic operations and the Archimedean property (Axiom 9.1). In Definition 8.10 we show how order automorphisms (strictly monotone functions) of R arise from *relations* rather than functions on Q.

Remark 6.7 There are many formulations of Dedekind cuts. We choose a two-sided version that appears as Exercise 5.3.3 in [TvD88], but we don't know who originally formulated it. That book, like may other accounts, uses a one-sided version in its main development.

As in the case of open and closed subspaces (Remark 2.6f), the relationship between the two sides of the cut is *contravariant*. One-sided accounts obtain one from the other by means of (Heyting) negation — with some awkwardness, since double negation is not the identity in intuitionistic logic.

The ASD calculus does not allow negation of predicates. Nor, indeed, any kind of contravariant operation, since they can neither be continuous nor computable. So we have to use both halves of the cut. But this is not just to overcome a technical handicap of our weak logic: like open and closed subspaces, the two halves of the cut play complementary roles that are plainly of equal importance.

We call δ or υ ***rounded*** if they satisfy the first or second of the following properties, these being the most important. Rounded pairs that fail one or more of the other conditions may be used to represent $+\infty \equiv (\top, \bot)$, $-\infty \equiv (\bot, \top)$ and intervals, as in Definition 3.4.

Definition 6.8 Formalising Remark 3.1, a (***Dedekind***) ***cut*** (δ, υ) in a (not necessarily discrete) dense linear order without endpoints $(Q, <)$ is a pair of predicates

$$\Gamma,\ q : Q\ \vdash\ \delta q,\ \upsilon q : \Sigma,$$

such that

$\Gamma,\ u : Q$	$\vdash$	$\upsilon u\ \Leftrightarrow\ \exists t{:}Q.\ \upsilon t\ \wedge\ (t < u)$	υ rounded upper
$\Gamma,\ d : Q$	$\vdash$	$\delta d\ \Leftrightarrow\ \exists e{:}Q.\ (d < e)\ \wedge\ \delta e$	δ rounded lower
Γ	$\vdash$	$\exists u{:}Q.\ \upsilon u\ \Leftrightarrow\ \top$	bounded above
Γ	$\vdash$	$\exists d{:}Q.\ \delta d\ \Leftrightarrow\ \top$	bounded below
$\Gamma,\ d, u : Q$	$\vdash$	$\delta d\ \wedge\ \upsilon u\ \Rightarrow\ (d < u)$	disjoint
$\Gamma,\ d, u : Q$	$\vdash$	$(d < u)\ \Rightarrow\ (\delta d\ \vee\ \upsilon u)$	located

We do not introduce the "set" of cuts, *cf.* Remark 5.1. In this section, this Definition is merely a property that a pair of predicates on Q may or may not have; in the next, we shall explain how it defines a new type R in the ASD λ-calculus.

There are rationals that straddle the cut but are arbitrarily close together. We shall prove a stronger version of this result in Proposition 9.3.

Lemma 6.9 Let $\Gamma \vdash q_0 < q_1 < \cdots < q_{n-1} : Q$ be a strictly ascending sequence of rationals. Then any cut (δ, υ) belongs to at least one of the *overlapping* open intervals

$$(-\infty, q_1),\quad (q_0, q_2),\quad (q_1, q_3),\quad \ldots,\quad (q_{n-3}, q_{n-1}),\quad (q_{n-2}, \infty)$$

in the sense that

$$\Gamma\ \vdash\ \upsilon q_1\ \vee\ \bigvee_{j=0}^{n-3} (\delta q_j \wedge \upsilon q_{j+2})\ \vee\ \delta q_{n-2}\ \Leftrightarrow\ \top.$$

Proof Apply the locatedness axiom to each pair $q_j < q_{j+1}$, giving

$$(\delta q_0 \vee \upsilon q_1)\ \wedge\ (\delta q_1 \vee \upsilon q_2)\ \wedge\ \cdots\ \wedge\ (\delta q_{n-2} \vee \upsilon q_{n-1}).$$

When we expand this using distributivity, most of the terms vanish by disjointness. Those that don't are of the form

$$(\delta q_0 \wedge \cdots \wedge \delta q_j)\ \wedge\ (\upsilon q_{j+2} \wedge \cdots \wedge \upsilon q_{n-1}),$$

including the cases with no δs and all υs or *vice versa*. Since in each case only the innermost conjuncts are needed, by roundedness, the expansion is as claimed. Note that this uses list induction, and relies on Q being overt discrete [E]. □

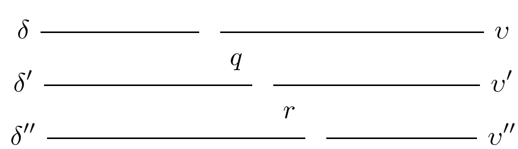
Definition 6.10 The irreflexive (strict) order relation on cuts is the *predicate*

$$(\delta, \upsilon) < (\delta', \upsilon')\ \equiv\ \exists q.\ \upsilon q \wedge \delta' q : \Sigma.$$

The idea is that the right part of the smaller cut overlaps the left part of the larger.

Lemma 6.11 The relation $<$ is transitive.

Proof



□

Lemma 6.12 As in Remark 3.1, each $q : Q$ gives rise to a cut (δ_q, υ_q) with

$$\delta_q \;\equiv\; \lambda d.\,(d < q) \qquad \text{and} \qquad \upsilon_q \;\equiv\; \lambda u.\,(q < u).$$

This is an order embedding in the sense that

$$(q < s) \;\Leftrightarrow\; (\exists r.\, q < r < s) \;\Leftrightarrow\; (\exists r.\, \upsilon_q r \wedge \delta_s r) \;\equiv\; \big((\delta_q, \upsilon_q) < (\delta_s, \upsilon_s)\big). \qquad \square$$

There is also a reflexive (non-strict) order $\leq$, given by implication (inclusion) between the cuts considered as predicates (subsets), so $\leq$ is a *statement*, *cf.* Remark 4.7.

Proposition 6.13 Let $\Gamma \;\vdash\; (\delta, \upsilon),\; (\delta', \upsilon')$ be cuts. Then the three statements

$$\delta \geq \delta' : \Sigma^Q, \qquad \upsilon \leq \upsilon' : \Sigma^Q \qquad \text{and} \qquad \big((\delta, \upsilon) < (\delta', \upsilon')\big) \;\Leftrightarrow\; \bot$$

are equivalent. We write $\Gamma \vdash (\delta', \upsilon') \leq (\delta, \upsilon)$ for any of them.

Proof Since $\big((\delta, \upsilon) < (\delta', \upsilon')\big)$ is $\exists q.\, \upsilon q \wedge \delta' q$, it is $\bot$ if either $\delta' \leq \delta$ or $\upsilon \leq \upsilon'$, by disjointness.

Conversely, by the definition of $\exists$, another way of writing the statement with $<$ is

$$\Gamma,\; q : Q \;\vdash\; (\upsilon q \wedge \delta' q) \;\Leftrightarrow\; \bot$$

Then

$$\begin{array}{rcll}
\Gamma,\; p : Q \;\vdash\; \delta' p & \Leftrightarrow & \exists q.\, (p < q) \wedge \delta' q & \delta \text{ is rounded} \\
& \Rightarrow & \exists q.\, (\delta p \vee \upsilon q) \wedge \delta' q & (\delta, \upsilon) \text{ located} \\
& \Leftrightarrow & \exists q.\, (\delta p \wedge \delta' q) \;\vee\; (\upsilon q \wedge \delta' q) & \text{distributivity} \\
& \Rightarrow & \exists q.\, (\delta p \vee \bot) \;\equiv\; \delta p & \text{hypothesis,}
\end{array}$$

and similarly $\upsilon p \Rightarrow \upsilon' p$. $\square$

Corollary 6.14 Let $\Gamma \;\vdash\; a \equiv (\delta, \upsilon),\; b \equiv (\delta', \upsilon')$ be cuts. Then

$$\frac{\Gamma \;\vdash\; a \;=\; b}{\Gamma \;\vdash\; \big((a < b) \;\vee\; (b < a)\big) \;\Leftrightarrow\; \bot : \Sigma}$$

From this we will deduce that R is Hausdorff and satisfies the trichotomy law (Definition 6.1(c)). However, in order to say that $<$ is interpolative and extrapolative, we must first define $\exists_R$, and so R itself as a type. We do that in the next section, and discuss Dedekind cuts of R in Section 8. $\square$

Exercise 6.15 Formulate the following in accordance with Remark 4.7, and prove them:
(a) if $a \leq b < c \leq d$ then $a < d$;
(b) if $a < b$ or $a = b$ then $a \leq b$;
(c) if $a \leq b$ and $a \neq b$ then $a < b$;
(d) if δd then $(d < u) \vee \delta u$;
(e) if $(d < u) \vee \delta u$ for all u then δd [NB: $\forall u : Q$ is not allowed];
(f) not all $d : Q$ satisfy δd.
Finally, verify that the results about $<$ and $\leq$ make no use of the boundedness axiom for cuts, and therefore also apply to $+\infty \equiv (\top, \bot)$ and $-\infty \equiv (\bot, \top)$. $\square$

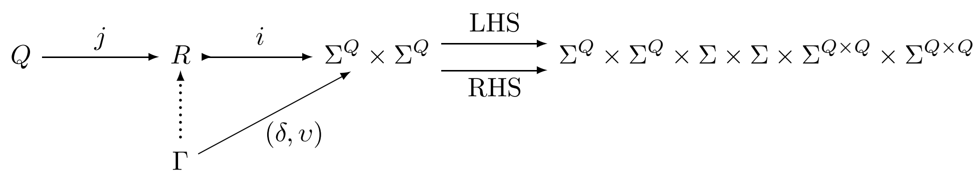
7 R as a S-split subspace

Now we put the ideas of Sections 3, 5 and 6 together, to define the space R of Dedekind cuts as a type in ASD. First we summarise what we achieved in the previous section.

Remark 7.1 The six axioms for a cut in Definition 6.8 are (essentially) equations (*cf.* Axiom 4.6). The boundedness axioms are already equations between terms of type Σ, whilst the roundedness conditions become equations of type Σ^Q when we λ-abstract the variable. Finally, the requirement that (δ, υ) be disjoint and located may be rewritten as

$$\begin{aligned} \Gamma \ \vdash \ \big(\lambda du.\, \delta d \wedge \upsilon u \wedge (d < u)\big) \quad &= \quad \big(\lambda du.\, \delta d \wedge \upsilon u\big) : \Sigma^{Q \times Q} \\ \Gamma \ \vdash \ \big(\lambda du.\, \delta d \vee \upsilon u \vee (d < u)\big) \quad &= \quad \big(\lambda du.\, \delta d \vee \upsilon u\big) : \Sigma^{Q \times Q}. \end{aligned}$$

We therefore expect R to be the subspace of $\Sigma^Q \times \Sigma^Q$ of those pairs (δ, υ) that satisfy the equations. In the language of category theory, this is the equaliser of the parallel pair of arrows that collects the six equations:



Definition 7.2 Recall that being the ***equaliser*** means that

(a) the inclusion i has equal composites with the parallel pair, *i.e.* it sends each $a : R$ to a pair $(\lambda d.\, d < a, \lambda u.\, a < u) : \Sigma^Q \times \Sigma^Q$ that satisfies the six equations; and

(b) whenever we have another map $\Gamma \to \Sigma^Q \times \Sigma^Q$ (which is the same as a pair of predicates $\Gamma,\ q : Q \vdash \delta q, \upsilon q : \Sigma$) that has equal composites with the parallel pair (*i.e.* which satisfies the six equations), then we may introduce a map $\Gamma \to R$ (term $\Gamma \vdash a : R$) making the triangles commute (*i.e.* $(\delta, \upsilon) = (\lambda d.\, d < a, \lambda u.\, a < u)$), and this is unique.

We have also drawn Q in this diagram, because Lemma 6.12 makes it an example of such a Γ. The map $Q \to \Sigma^Q \times \Sigma^Q$ by $q \mapsto (\delta_q, \upsilon_q)$ is mono (1–1) by Lemma 6.12, and therefore $j : Q \to R$ is mono too.

However, this equaliser still only tells us about the "points" (more accurately, terms) of R, not its topology. As Examples 12.2 and 12.6 will show, the equaliser exists in many categories, but may yield a closed interval that fails to be compact. But the observations in Section 3 showed how compactness of the closed interval can be built into the definition of R, by making Σ^R a Scott-continuous retract of $\Sigma^{\Sigma^Q \times \Sigma^Q}$.

Notation 7.3 Now we recall the "interesting formula" from Proposition 3.8, an idempotent $\mathcal{E}$ on $\Sigma^{\Sigma^Q \times \Sigma^Q}$ that was defined by

$$\mathcal{E}\Phi(\delta, \upsilon) \ \equiv \ \exists q_0 < \cdots < q_{2n+1}.\ \delta q_1 \wedge \upsilon q_{2n} \ \wedge \ \bigwedge_{k=0}^{n-1} \Phi(\lambda x.\, x < q_{2k},\ \lambda x.\, q_{2k+3} < x),$$

where there is *at least one* conjunct ($n \geq 1$).

As this formula only involves variables ranging over Q, Σ^Q and $\Sigma^{\Sigma^Q \times \Sigma^Q}$, we may import it into the formalism of ASD. We draw attention to one formal point: "$\exists q_0 < \cdots < q_{2n+1}$" is existential quantification over $\mathsf{List}(Q)$. The ordering of the list is an open predicate that, like the n-fold conjunction, is easily definable by recursion on lists. The construction of $\mathsf{List}(Q)$ in ASD is set out in [E], and requires Q to be overt discrete.

Remark 7.4 Recall from Section 5 that we define Σ-split subspaces in the *abstract* calculus of ASD from a nucleus, *i.e.* a formula $\mathcal{E}$ satisfying the two equations below. The calculus then provides the map I, without relying on a proof of a result such as Proposition 3.7 in another structure. So far, therefore, we are *only* considering $\mathcal{E}$ *as an interesting formula*, and do not, for example, assume the classical Heine–Borel theorem. We therefore have to prove the relevant properties of $\mathcal{E}$ again from scratch. We shall find in Section 11 that $\mathcal{E}$ is a *very* interesting formula, providing both quantifiers over the closed interval in the cases where δ and υ are disjoint ($\forall$) or overlap ($\exists$).

Proposition 7.5 $\mathcal{E}$ is a nucleus (Definition 5.6), as it satisfies

$$\Phi, \Psi : \Sigma^{\Sigma^Q \times \Sigma^Q} \ \vdash\ \mathcal{E}(\Phi \wedge \Psi) \ =\ \mathcal{E}(\mathcal{E}\Phi \wedge \mathcal{E}\Psi) \quad \text{and} \quad \mathcal{E}(\Phi \vee \Psi) \ =\ \mathcal{E}(\mathcal{E}\Phi \vee \mathcal{E}\Psi). \qquad \square$$

This fulfils the obligation that we have before invoking the ASD technology in Section 5.

Corollary 7.6 In ASD there is a type $R \equiv \{\Sigma^Q \times \Sigma^Q \mid \mathcal{E}\}$ together with (formally defined) inclusions

$$i : R \rightarrowtail \Sigma^Q \times \Sigma^Q \quad \text{and} \quad I : \Sigma^R \rightarrowtail \Sigma^{\Sigma^Q \times \Sigma^Q}$$

such that $I\phi(ix) = \phi x$ and $\mathcal{E} = \Sigma^i \cdot I$. $\square$

Remark 7.7 The operators admit, i and I in the formal calculus (Axioms 5.8–5.9) ensure that every term has a unique type. However, the distinction between cuts and reals is clear enough (in so far as it is necessary) in this paper, so we follow the usual mathematical custom by omitting admit and i. There is, as we intended in Remark 3.3, also a similar association between $\phi : \Sigma^R$ and $\Phi : \Sigma^{\Sigma^Q \times \Sigma^Q}$, given by

$$\Phi \ \equiv\ I\phi \quad \text{and} \quad \phi \equiv \Sigma^i \Phi \equiv \lambda x.\, \Phi(\delta_x, \upsilon_x),$$

but we shall usually take this for granted too.

We now have two different notions of "real number" in ASD — Dedekind cuts from the previous section, and admissible terms with respect to $\mathcal{E}$. Definition 5.8 said that the map i is the equaliser of a particular pair of maps involving $\mathcal{E}$. It is the business of the rest of this section to show that i is also the equaliser of the parallel pair of maps in Remark 7.1, *i.e.* that the two notions of real number are the same.

Before we can do this, we have to reproduce part of the proof of Proposition 3.8, deducing a fundamental property of the "Euclidean" topology of the real line from the general Scott continuity (Axiom 4.20) of λ-terms of higher type.

Lemma 7.8 If $\Gamma \vdash (\delta, \upsilon)$ are rounded and bounded then the joins

$$\delta \ =\ \exists d.\, \delta d \wedge (\lambda x.\, x < d) \quad \text{and} \quad \upsilon \ =\ \exists u.\, \upsilon u \wedge (\lambda x.\, u < x)$$

are *directed* in the sense of Definition 4.19.

Proof Boundedness gives (a). The binary operation @ in (b) is max for δ and min for υ. $\square$

Proposition 7.9 Any $\Gamma \vdash \Phi : \Sigma^{\Sigma^Q \times \Sigma^Q}$ is ***rounded*** in the sense that

$$\text{if } (\delta, \upsilon) \text{ are rounded and bounded then} \quad \Phi(\delta, \upsilon) \ \Leftrightarrow\ \exists du.\, \delta d \wedge \upsilon u \wedge \Phi(\delta_d, \upsilon_u). \qquad \square$$

Corollary 7.10 Any property Φ that is true of some real number (*i.e.* a cut $a \equiv (\delta, \upsilon)$), is also true of some rational numbers either side of it ($e < a < t$):

$$\Phi(\delta, \upsilon) \ \Rightarrow\ \exists et{:}Q.\ \Phi(\delta_e, \upsilon_e) \wedge \delta e \wedge \upsilon t \wedge \Phi(\delta_t, \upsilon_t).$$

Proof If $d < e < t < u$ then $\delta_d \leq \delta_e$ and $\upsilon_u \leq \upsilon_t$. $\square$

Lemma 7.11 If (δ, υ) is a cut then it is admissible.

Proof Applying Lemma 6.9 to $q_0 < \cdots < q_{2n+1}$ in Notation 7.3, we have

$$\exists q_0 \ldots q_{2n+1}.\ \delta q_1 \wedge \big(\upsilon q_1 \vee \bigvee_{j=1}^{2n-2} (\delta q_j \wedge \upsilon q_{j+2}) \vee \delta q_{2n}\big) \wedge \upsilon q_{2n} \wedge \bigwedge_{k=0}^{n-1} \Phi(\delta_{q_{2k}}, \upsilon_{q_{2k+3}}),$$

so

$$\begin{aligned} \mathcal{E}\Phi(\delta,\upsilon) &\Rightarrow \exists q_0 \ldots q_{2n+1}.\ \bigvee_{k=0}^{n-1} \big(\delta q_{2k} \wedge \upsilon q_{2k+3} \wedge \Phi(\delta_{q_{2k}}, \upsilon_{q_{2k+3}})\big) \\ &\Rightarrow \exists d < u.\ \delta d \wedge \upsilon u \wedge \Phi(\delta_d, \upsilon_u) \Leftrightarrow \Phi(\delta, \upsilon) \qquad d \equiv q_{2k},\ u \equiv q_{2k+3} \\ &\Rightarrow \exists q_0 < \cdots < q_3.\ \delta q_1 \wedge \upsilon q_2 \wedge \Phi(\delta_{q_0}, \upsilon_{q_3}) \Rightarrow \mathcal{E}\Phi(\delta,\upsilon), \end{aligned}$$

i.e. a single interval $[q_1, q_2] \subset (q_0, q_3)$ would have been enough for the expansion. The argument is completed by Proposition 7.9 and roundedness. □

Lemma 7.12 If (δ, υ) is admissible then it is a cut.

Proof We have to deduce each of the parts of Definition 6.8 from instances of admissibility, $\mathcal{E}\Phi(\delta,\upsilon) \Leftrightarrow \Phi(\delta,\upsilon)$, with respect to carefully chosen Φ.

For *boundedness*, consider $\Phi \equiv \lambda\alpha\beta.\ \top$. Then

$$\top \equiv \Phi(\delta,\upsilon) \Leftrightarrow \mathcal{E}\Phi(\delta,\upsilon) \equiv \exists q_0 < \cdots < q_{2n+1}.\ \delta q_1 \wedge \upsilon q_{2n}.$$

For *roundedness*, consider $\Phi \equiv \lambda\alpha\beta.\ \alpha d \wedge \beta u$. Then $\delta d \wedge \upsilon u \equiv \Phi(\delta,\upsilon) \Leftrightarrow \mathcal{E}\Phi(\delta,\upsilon)$, whose expansion is

$$\exists q_0 < \cdots < q_3.\ \delta q_1 \wedge \upsilon q_2 \wedge (d < q_0) \wedge (q_3 < u) \Leftrightarrow \exists q_1 q_2.\ (d < q_1) \wedge \delta q_1 \wedge (q_2 < u) \wedge \upsilon q_2.$$

For *disjointness*, consider $\Phi \equiv \lambda\alpha\beta.\ \bot$. Then the big conjunction is either empty $(n = 0)$ or false $(n \geq 1)$, so

$$\bot \equiv \Phi(\delta,\upsilon) \Leftrightarrow \mathcal{E}\Phi(\delta,\upsilon) \equiv \exists q_0 < \cdots < q_{2n+1}.\ \delta q_1 \wedge \upsilon q_{2n} \wedge (n = 0) \Leftrightarrow \exists q_0 < q_1.\ \delta q_1 \wedge \upsilon q_0.$$

For *locatedness*, let $d < u$. Since (δ, υ) are rounded and bounded, there is a sequence of rationals

$$q_0 < q_1 < d < q_2 < q_3 < u < q_4 < q_5 \quad \text{with} \quad \delta q_1 \quad \text{and} \quad \upsilon q_4.$$

Then $\Phi(\alpha,\beta) \equiv \alpha d \vee \beta u$ gives $\delta d \vee \upsilon u \equiv \Phi(\delta,\upsilon) \Leftrightarrow \mathcal{E}\Phi(\delta,\upsilon)$, whose expansion is implied by

$$(q_0 < \cdots < q_5) \wedge \delta q_1 \wedge \upsilon q_4 \wedge (d < q_0 \vee q_3 < u) \wedge (d < q_2 \vee q_5 < u). \qquad \square$$

Corollary 7.13 The type R defined in Corollary 7.6 is the equaliser in Remark 7.1. □

In Section 11 we show that this construction really does make the closed interval compact.

8 Dedekind completeness

Now we show that any Dedekind cut of R corresponds to an element of R, and not of a yet more complicated structure. However, in order to say what a "Dedekind cut of R" is, we must finish the proof that its order (Definition 6.10) is dense without endpoints, *i.e.* satisfies Definition 6.1, which in turn requires that the object R be Hausdorff (Corollary 6.14) and overt.

Proposition 8.1 R is overt, with ***existential quantifier***

$$\frac{\Gamma,\ x : R \vdash \phi x \Rightarrow \sigma}{\Gamma \vdash \exists x{:}R.\ \phi x \Rightarrow \sigma}$$

where

$$\exists r{:}R.\ \phi r \equiv \exists q{:}Q.\ \phi q \Leftrightarrow \exists q{:}Q.\ (I\phi)(\delta_q, \upsilon_q) \Leftrightarrow (I\phi)(\top, \top).$$

Proof As Q is overt, we already have the rule with Q in place of R. The top line (for R) entails the similar judgement for Q by restriction, and then this entails the common bottom line. Conversely, by Corollary 7.10, $\Gamma,\ x:R \vdash \phi x \Rightarrow \exists q.\,\phi q \Rightarrow \sigma$.

It remains to justify the formula $(I\phi)(\top,\top)$. When we substitute $\delta\equiv\upsilon\equiv\top$ in Notation 7.3, it still has at least one conjunct, and in fact that is enough:

$$\begin{array}{rcll}\mathcal{E}\Phi(\top,\top) & \Leftrightarrow & \exists q_0<\cdots<q_{2n+1}.\ \bigwedge_{k=0}^{n-1}\Phi(\delta_{q_{2k}},\upsilon_{q_{2k+3}}) & \\ & \Rightarrow & \exists d<u.\ \Phi(\delta_d,\upsilon_u) & d\equiv q_{2k},\ u\equiv q_{2k+3} \\ & \Rightarrow & \exists q.\ \Phi(\delta_q,\upsilon_q) & \text{any } d\le q\le u \\ & \Rightarrow & \Phi(\top,\top), & \delta_q\le\top,\ \upsilon_q\le\top\end{array}$$

in which we put $\Phi\equiv I\phi = I\Sigma^i I\phi = \mathcal{E}\Phi$. □

Proposition 8.2 $(R,<)$ is a dense linear order without endpoints.

Proof We already know that the order is transitive. It is extrapolative and interpolative in the sense that

$$\top \Leftrightarrow \exists q{:}Q.\ (\delta_q,\upsilon_q)<(\delta,\upsilon)$$

and

$$(\delta,\upsilon)<(\delta'',\upsilon'') \quad\Rightarrow\quad \exists r{:}Q.\ (\delta,\upsilon)<(\delta_r,\upsilon_r)<(\delta'',\upsilon''),$$

but existential quantification over Q may be replaced by that over R. □

Now we can define Dedekind cuts over R, but before we do so, let's state the correspondence between Q-cuts and R-cuts simply using their underlying λ-terms.

Lemma 8.3 The maps $(\delta,\upsilon):\Sigma^Q\times\Sigma^Q \rightleftarrows \Sigma^{\Sigma^Q\times\Sigma^Q}\times\Sigma^{\Sigma^Q\times\Sigma^Q} : (\Delta,\Upsilon)$ given by

$$\begin{array}{rcl}(\delta,\upsilon) & \mapsto & (\lambda\alpha\beta.\ \exists d.\ \beta d\wedge\delta d,\ \lambda\alpha\beta.\ \exists u.\ \upsilon u\wedge\alpha u) \\ (\Delta,\Upsilon) & \mapsto & \big(\lambda d.\ \Delta(\delta_d,\upsilon_d),\ \lambda u.\ \Upsilon(\delta_u,\upsilon_u)\big)\end{array}$$

define a bijection between rounded (δ,υ) and those "canonical" (Δ,Υ) that satisfy

$$\Delta(\alpha,\beta) \Leftrightarrow \exists d.\ \beta d\wedge\Delta(\delta_d,\upsilon_d) \quad\text{and}\quad \Upsilon(\alpha,\beta) \Leftrightarrow \exists u.\ \Upsilon(\delta_u,\upsilon_u)\wedge\alpha u$$

for *all* $\alpha,\beta:\Sigma^Q$, not just cuts. Notice that then Δ actually only uses its second argument and Υ its first. □

Lemma 8.4 The same conditions, but restricted to *cuts* (α,β), are equivalent to roundedness of $(\Delta,\Upsilon):\Sigma^R\times\Sigma^R$:

$$\begin{array}{rcl}\exists q.\ \big((\alpha,\beta)<(\delta_q,\upsilon_q)\big)\ \wedge\ \Delta(\delta_q,\upsilon_q) & \Leftrightarrow & \Delta(\alpha,\beta) \\ \exists q.\ \Upsilon(\delta_q,\upsilon_q)\ \wedge\ \big((\delta_q,\upsilon_q)<(\alpha,\beta)\big) & \Leftrightarrow & \Upsilon(\alpha,\beta).\end{array}$$

If (Δ,Υ) satisfy these weaker properties, then we still obtain rounded (δ,υ) from them by the formula above. From the latter we derive *canonical* (Δ',Υ') that agree with the given (Δ,Υ) *when applied to cuts.* □

Lemma 8.5 Under this correspondence, (δ,υ) are disjoint, rounded or located iff (Δ,Υ) have the same property.

Proof These properties may be expressed either as existentially quantified statements (over Q or R), or as conditional properties of cuts. These equivalent triplets are as follows:

bounded: $(\exists d.\ \delta d)\wedge(\exists u.\ \upsilon u)\Leftrightarrow\top$ $\big(\exists d.\ \Delta(\delta_d,\upsilon_d)\big)\wedge\big(\exists u.\ \Upsilon(\delta_u,\upsilon_u)\big)\Leftrightarrow\top$

$\Delta(\alpha,\beta)\wedge\Upsilon(\sigma,\tau) \Leftrightarrow \top$ for some cuts (α,β) and (σ,τ)

disjoint: $(\delta d\wedge\upsilon u)\Rightarrow(d<u)$ $\Delta(\delta_d,\upsilon_d)\wedge\Upsilon(\delta_u,\upsilon_u) \Rightarrow (d<u)$

$\Delta(\alpha,\beta)\wedge\Upsilon(\sigma,\tau) \Rightarrow \big((\alpha,\beta)<(\sigma,\tau)\big)$

located: $(d<u)\Rightarrow(\delta d\vee\upsilon u)$ $(d<u) \Rightarrow \Delta(\delta_d,\upsilon_d)\vee\Upsilon(\delta_u,\upsilon_u)$

$\Delta(\alpha,\beta)\vee\Upsilon(\sigma,\tau)$ whenever $(\alpha,\beta)<(\sigma,\tau)$ are cuts. □

Remark 8.6 The next stage in the definition of R from Q was the formulation of $\mathcal{E}$, *i.e.* what we did in the previous section. However, we cannot repeat this step with R in place of Q, because existential quantification over finite lists in R is not allowed in Notation 7.3.

Why on earth not? This has nothing to do with rational or irrational numbers: so far the elements of Q are only nominally rational, as we haven't introduced any arithmetic yet. The point is topological: Q is overt discrete — essentially a "set" — and $\mathsf{List}(Q)$ is the overt discrete "hyperspace" of its overt compact subspaces [E]. On the other hand, R is not discrete but Hausdorff, and its overt compact subspaces can be infinite — bounded nontrivial closed intervals, for example. In the hyperspace of such subspaces, the finite ones are dense, so there is no way to distinguish finite from infinite ones.

But this doesn't matter. We don't need to define another space $S \rightarrowtail \Sigma^R \times \Sigma^R$ from such an $\mathcal{E}$, because we have already done enough to prove completeness, *i.e.* that R itself serves for S. Then $R \rightarrowtail \Sigma^R \times \Sigma$ will be Σ-split by the same argument as in Remark 3.7 (the rest of the section being redundant), as we will be able to express this in ASD after we have proved compactness of $[d,u]$.

Theorem 8.7 $(R,<)$ is a ***complete*** dense linear order without endpoints, in the sense that every cut $\Gamma \vdash (\Delta,\Upsilon) : \Sigma^R \times \Sigma^R$ is of the form

$$\Gamma \vdash \Delta = \lambda x.\,(x<a) \quad\text{and}\quad \Upsilon = \lambda y.\,(a<y)$$

for some unique $\Gamma \vdash a : R$.

Proof In the previous section we defined Σ^R as a retract of $\Sigma^{\Sigma^Q \times \Sigma^Q}$, so we treat Δ and Υ as the restrictions to Σ^R of terms of type $\Sigma^{\Sigma^Q \times \Sigma^Q}$. From these, Lemma 8.3 defined $(\delta,\upsilon) : \Sigma^Q \times \Sigma^Q$ by

$$\delta d \equiv \Delta(jd) \equiv \Delta(\delta_d,\upsilon_d) \quad\text{and}\quad \upsilon u \equiv \Upsilon(ju) \equiv \Upsilon(\delta_u,\upsilon_u),$$

from which we obtain $\Delta'(\alpha,\beta) \equiv \exists d.\,\beta d \wedge \delta d$ and $\Upsilon'(\sigma,\tau) \equiv \exists u.\,\upsilon d \wedge \sigma d$.

Since (Δ,Υ) was rounded, (Δ',Υ') is canonical by Lemma 8.4, and we recover (δ,υ) from it. By Lemma 8.5, this satisfies the properties of a Q-cut because (Δ,Υ) had them as an R-cut. Therefore (δ,υ) corresponds to some $a : R$. By Lemma 8.4 again, (Δ',Υ') and (Δ,Υ) agree on cuts $x \equiv (\alpha,\beta)$ and $y \equiv (\sigma,\tau)$, so

$$\Delta x \equiv \Delta(\alpha,\beta) \Leftrightarrow \Delta'(\alpha,\beta) \equiv \exists d.\,\beta d \wedge \delta d \equiv (x<a)$$

and

$$\Upsilon y \equiv \Upsilon(\sigma,\tau) \Leftrightarrow \Upsilon'(\sigma,\tau) \equiv \exists u.\,\upsilon u \wedge \sigma u \equiv (a<y). \qquad \square$$

Two other important notions of completeness for the real line are considered in [J], namely the convergence of Cauchy sequences and the existence of suprema of nonempty bounded subsets. In any form of constructive analysis, the latter result requires additional hypotheses; in ASD these are that the subset be compact and overt.

A natural special case of the supremum is the maximum of a pair:

Proposition 8.8 R has binary meet and join with respect to $\le$,

$$\begin{aligned}\mathsf{max}\big((\delta_1,\upsilon_1),\ (\delta_2,\upsilon_2)\big) &\equiv (\delta_1 \vee \delta_2,\ \upsilon_1 \wedge \upsilon_2)\\ \mathsf{min}\big((\delta_1,\upsilon_1),\ (\delta_2,\upsilon_2)\big) &\equiv (\delta_1 \wedge \delta_2,\ \upsilon_1 \vee \upsilon_2).\end{aligned}$$

satisfying $\mathsf{min}(a,b) \le a, b \le \mathsf{max}(a,b)$ and

$$\begin{array}{rcl@{\qquad}rcl}(a<x)\wedge(b<x) & \Leftrightarrow & \mathsf{max}(a,b)<x & (a>x)\wedge(b>x) & \Leftrightarrow & \mathsf{min}(a,b)>x\\ (a>x)\vee(b>x) & \Leftrightarrow & \mathsf{max}(a,b)>x & (a<x)\vee(b<x) & \Leftrightarrow & \mathsf{min}(a,b)<x\\ (a\le x),\ (b\le x) & \dashv\vdash & \mathsf{max}(a,b)\le x & (a\ge x),\ (b\ge x) & \dashv\vdash & \mathsf{min}(a,b)\ge x.\end{array} \qquad \square$$

We have said that the two halves of a Dedekind cut play symmetrical roles. Adding an operation that makes this literally so, we have the beginnings of arithmetic:

Proposition 8.9 If $(Q, <)$ has an order-reversing automorphism $(-)$ then so does R, and this has a unique fixed point (0):

$$-(\delta, \upsilon) \equiv \big(\lambda d.\, \upsilon(-d),\ \lambda u.\, \delta(-u)\big) \qquad 0 \equiv (\lambda d.\, d < -d,\ \lambda u.\, -u < u).$$

Then there is an absolute value function,

$$|(\delta, \upsilon)| \equiv \mathsf{max}\big((\delta, \upsilon),\ -(\delta, \upsilon)\big) = \big(\lambda d.\, \delta d \vee \upsilon(-d),\ \lambda u.\, \upsilon u \wedge \delta(-u)\big),$$

so $|x| = x$ if $x \geq 0$ and $|x| = -x$ if $x \leq 0$. This also satisfies $(x \neq 0) \equiv (\delta 0 \vee \upsilon 0) \Rightarrow |x| \neq 0$. □

Of course, the main arithmetic operations ($+$ and $\times$) present us with the reasons for introducing Dedekind cuts in the first place, namely the solution of (algebraic and other) equations. A simple but important requirement is the inverse of a strictly monotone function, for example to find cube roots. However, it is more natural to present the technique symmetrically in the given function and its inverse, and then $x \mapsto x^{5/3}$ is an equally simple example. A particular problem of this kind is then formulated as a pair of binary *relations*. Anyone with a little knowledge of (lattice or) category theory will recognise this situation as an *adjunction*, or rather two of them.

Definition 8.10 A ***strictly monotone graph*** $Q_1 \leftrightarrow Q_2$ between dense linear orders without endpoints (Definition 6.1) is a pair $(\lessdot, \gtrdot)$ of binary relations satisfying

$$\begin{array}{rcccl} \exists b.\, a <_1 b \lessdot x & \Leftrightarrow & a \lessdot x & \Leftrightarrow & \exists y.\, a \lessdot y <_2 x \\ \exists b.\, x \gtrdot b <_1 a & \Leftrightarrow & x \gtrdot a & \Leftrightarrow & \exists y.\, x <_2 y \gtrdot a \\ \exists x.\, a \lessdot x \gtrdot b & \Leftrightarrow & a <_1 b \qquad x <_2 y & \Leftrightarrow & \exists a.\, x \gtrdot a \lessdot y \end{array}$$

where $a, b : Q_1$ and $x, y : Q_2$. These objects could, for example, consist of just *positive* rationals, or carry the opposite of the usual order. Let R_1 and R_2 be their Dedekind completions.

Proposition 8.11 Any strictly monotone graph defines inverse functions $R_1 \cong R_2$ by

$$\begin{array}{rcl} f(\delta_1, \upsilon_1) & \equiv & (\lambda d_2.\, \exists d_1.\, d_2 \lessdot d_1 \wedge \delta_1 d_1,\ \lambda u_2.\, \exists u_1.\, u_1 \gtrdot u_2 \wedge \upsilon_1 u_1) \\ g(\delta_2, \upsilon_2) & \equiv & (\lambda d_1.\, \exists d_2.\, d_1 \gtrdot d_2 \wedge \delta_2 d_2,\ \lambda u_1.\, \exists u_2.\, u_2 \lessdot u_1 \wedge \upsilon_2 u_2). \end{array}$$

Proof We have to show that these formulae take Dedekind cuts to Dedekind cuts. The first four axioms make $f(\delta_1, \upsilon_1)$ and $g(\delta_2, \upsilon_2)$ rounded. Then we use the last two axioms to show that, if (δ_1, υ_1) and (δ_2, υ_2) are cuts, so are $f(\delta_1, \upsilon_1)$ and $g(\delta_2, \upsilon_2)$, and that these maps are inverse. □

9 Arithmetic

Real arithmetic is a clear example of familiarity breeding contempt: we do not know of a published account that gives the details in a form that is suitable for us. Dedekind himself considered addition and square roots but not multiplication [Ded72]. Even in the classical situation, the proof that the *positive and negative* reals form a commutative ring or field involves either an explosion of cases or considerable ingenuity. Case splitting, at least for R, is unacceptable in constructive analysis [TvD88, §5.6], topos theory [Joh77, §6.6] or ASD. One ingenious solution is John Conway's definition of multiplication [Con76, pp.19–20]. We shall, in fact, use a new formula for multiplication (not Conway's) that almost eliminates case splitting altogether. General continuity will relieve us of the burden of checking the arithmetic laws.

Axiom 9.1 Q is an ordered commutative ring that obeys the ***Archimedean principle***,

$$p, q : Q \vdash q > 0 \Rightarrow \exists n{:}\mathbb{Z}.\, q(n-1) < p < q(n+1),$$

and has ***approximate division***,

$$d, q, u : Q \ \vdash \ (d < u) \ \wedge \ (q \neq 0) \ \Rightarrow \ \exists m{:}Q.\ (d < mq < u),$$

that is ***rounded*** in the sense that

$$d, m, q, u : Q \ \vdash \ (q > 0) \ \wedge \ (d < mq < u) \ \Rightarrow \ \exists et{:}Q.\ (d < eq) \wedge (e < m < t) \wedge (qt < u).$$

In view of Proposition 6.5, the Archimedean principle cannot be a property of the *order* $(Q, <)$ alone: its says how $\mathbb{Z}$ (the additive subgroup generated by q) lies within the order. If, as we intend, Q is discrete, Corollary 7.10 does not make roundedness redundant.

Remark 9.2 We don't ask that Q be a *field*, *i.e.* with exact division, because we have in mind the ways in which it might be represented computationally: the axioms ought at least to allow the dyadic and decimal rationals as examples. Notice, however, that we use the arithmetic operations on Q below only in the form of the four ternary relations

$$a < b + c \qquad u > s + t \qquad a < b \times c \qquad u > s \times t,$$

whose axiomatisation we leave to the interested reader. Such a person may be an interval analyst who wants to use ASD to obtain mathematically provable (albeit incomplete) results from the ***floating point arithmetic*** that is built in to computer hardware.

In hardware arithmetic, not even addition and multiplication (let alone division and the trigonometrical functions) are exact. The strict operations in the following development can be replaced by downward- and upward-rounded versions, for which code libraries are widely available. The incompleteness manifests itself in our context as the failure of locatedness, so the modified operations act on intervals *à la* Moore (Definition 3.4) instead of on numbers or Dedekind cuts.

On the subject of locatedness, we shall need a stronger form of it for arithmetic than the order-theoretic formulation that we stated in Definition 6.8 ($d < u \Rightarrow \delta d \vee \upsilon u$). This is actually the version that most accounts of Dedekind cuts use.

Proposition 9.3 $(\delta, \upsilon) : R,\ e : Q \ \vdash \ e > 0 \ \Rightarrow \ \exists d, u{:}Q.\ \delta d \wedge \upsilon u \wedge (0 < u - d < e)$.

Proof By boundedness, $\delta d_0 \wedge \upsilon u_0$ for some $d_0 < u_0 : Q$. By approximate division and the Archimedean principle (Axiom 9.1), $0 < 2e' < e$ and $u_0 - d_0 < ne'$ for some $e' : Q$ and $n : \mathbb{N}$. Then $q_j \equiv d_0 + je'$ defines a sequence

$$q_{-1} < q_0 \equiv d_0 < q_1 < q_2 < \cdots < q_{n-1} < q_n > u_0.$$

to which we may apply Lemma 6.9. This gives $\delta d \wedge \upsilon u \wedge (u - d < e)$, where $d \equiv q_{j-1}$, $u \equiv q_{j+1}$ and $u - d = 2e' < e$, for some $0 \le j \le n$. ☐

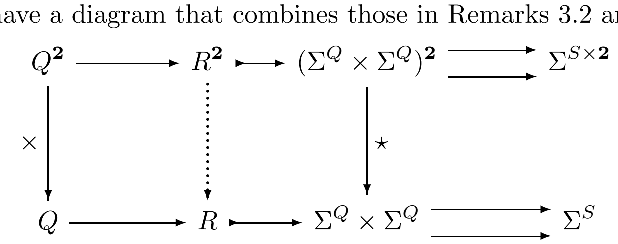
Since the arithmetic operations involve binary operations and ternary laws, we have to define the objects $R^{\mathbf{2}}$ and $R^{\mathbf{3}}$ before we can extend the operations and laws from Q to R.

Proposition 9.4 The product spaces $R^{\mathbf{2}}$, $R^{\mathbf{3}}$, *etc.* exist, they are overt, and the inclusions $Q^{\mathbf{n}} \to R^{\mathbf{n}}$ are dense.

Proof Axiom 4.1 gives product types, and the quantifiers are given by

$$\begin{aligned} \exists (x, y){:}R^{\mathbf{2}}.\ \phi(x, y) \quad &\equiv \quad \exists x{:}R.\ \exists y{:}R.\ \phi(x, y) \\ &\Leftrightarrow \quad \exists x{:}Q.\ \exists y{:}Q.\ \phi(x, y) \ \Leftrightarrow \ \exists (x, y){:}Q^{\mathbf{2}}.\ \phi(x, y). \end{aligned}$$ ☐

Remark 9.5 Now we have a diagram that combines those in Remarks 3.2 and 7.1,



where $\rightarrowtail$ denotes an equaliser and $S \equiv Q + Q + \mathbf{1} + \mathbf{1} + Q \times Q + Q \times Q$.

This means that a term of type $R^{\mathbf{2}}$ is a pair of solutions of equations, defining a pair of cuts or a "Dedekind cross-hair", *i.e.* a division of the plane into quadrants. $R^{\mathbf{2}}$ is, in fact, a Σ-split subspace [B], though the splitting I_2 is not simply the square of that for R. But that doesn't matter, as we shall not need it anyway.

Exercise 9.6 By analogy with Proposition 3.7, find $I_2 : \Sigma^{R\times R} \rightarrowtail \Sigma^{\Sigma^Q\times\Sigma^Q\times\Sigma^Q\times\Sigma^Q}$.

Remark 9.7 Next, for each arithmetic operation ($+$, $\times$, *etc.*),

(a) we must first define some operation $\star$ on $\Sigma^Q \times \Sigma^Q$ that extends the given operation $\times$ on the rationals, in the sense that the rectangle above commutes, *i.e.*

$$q, r : Q \ \vdash \ (\delta_q, \upsilon_q) \star (\delta_r, \upsilon_r) \ = \ (\delta_{q\times r}, \upsilon_{q\times r}) : \Sigma^Q \times \Sigma^Q;$$

(b) then we have to show that $(\delta_1, \upsilon_1) \star (\delta_2, \upsilon_2)$ is a Dedekind cut whenever (δ_1, υ_1) and (δ_2, υ_2) are (so the composites $R^{\mathbf{2}} \rightrightarrows \Sigma^S$ are equal), thereby filling in the dotted arrow above;

(c) and finally verify the usual laws of arithmetic for the extended operations.

Since R is a space and not a set with decidable equality, we cannot use case analysis (on values in R) to do any of these things. In the case of multiplication, all three parts are quite a challenge. For (b), we have to check that each formula that we introduce for an operation actually gives rounded, bounded and located pairs (δ, υ).

Roundedness and boundedness typically follow from the *form* of the expressions that we use for the extended arithmetic operations, together with transitivity and interpolation, and extrapolation, respectively, for the order on Q. We get disjointness for free:

Lemma 9.8 Let $\delta, \upsilon : R^{\mathbf{n}} \rightrightarrows \Sigma^Q$ be "disjoint" on $Q^{\mathbf{n}}$, in the sense that

$$d, u : Q,\ \vec{q} : Q^{\mathbf{n}} \ \vdash \ \delta\vec{q}d \wedge \upsilon\vec{q}u \ \Rightarrow \ (d < u).$$

Then they are also disjoint on $R^{\mathbf{n}}$.

Proof By Proposition 9.4, $\delta\vec{x}d \wedge \upsilon\vec{x}u \ \Rightarrow \ \exists\vec{q}.\, \delta\vec{q}d \wedge \upsilon\vec{q}u \ \Rightarrow \ (d < u)$. □

On the other hand, topology has its advantages, particularly in ASD, where all definable maps are automatically continuous. Once we've thought of a definition for the extended operations, the laws of arithmetic (both equations and inequalities) transfer automatically from $Q^{\mathbf{n}}$ to $R^{\mathbf{n}}$. (For a metrical way of doing this, see [TvD88, p260].)

Lemma 9.9 For $f, g : R^{\mathbf{n}} \rightrightarrows R$,

$$\text{if} \quad \vec{q} : Q^{\mathbf{n}} \vdash f(\vec{q}) \leq g(\vec{q}) : R \quad \text{then} \quad \vec{x} : R^{\mathbf{n}} \vdash f(\vec{x}) \leq g(\vec{x}) : R.$$

Proof The traditional argument is this: since R is Hausdorff, the subspace

$$C \ \equiv \ \{\vec{x} : R^{\mathbf{n}} \mid f(\vec{x}) \leq g(\vec{x})\} \ \subset \ R^{\mathbf{n}}$$

on which the laws actually hold is closed; on the other hand, $Q^{\mathbf{n}} \subset R^{\mathbf{n}}$ is dense, whilst $Q^{\mathbf{n}} \subset C$ by hypothesis, so $C = R^{\mathbf{n}}$. In ASD, C is co-classified by ψ, where

$$\psi \ \equiv \ \lambda\vec{x}.\, f(\vec{x}) > g(\vec{x}), \quad \text{and} \quad \psi(\vec{x}) \ \Rightarrow \ \exists\vec{q}.\, \psi(\vec{q}) \ \Leftrightarrow \ \bot$$

by Proposition 9.4, so $\psi = \bot$. □

We are now ready to apply the plan in Remark 9.7 to addition.

Notation 9.10 For $(\delta_1, \upsilon_1), (\delta_2, \upsilon_2) : \Sigma^Q \times \Sigma^Q$, let

$$\begin{aligned}(\delta_1, \upsilon_1) + (\delta_2, \upsilon_2) \quad \equiv \quad & \big(\lambda d.\, \exists d_1 d_2.\, (d < d_1 + d_2) \wedge \delta_1 d_1 \wedge \delta_2 d_2, \\ & \ \lambda u.\, \exists u_1 u_2.\, (u_1 + u_2 < u) \wedge \upsilon_1 u_1 \wedge \upsilon_2 u_2\big).\end{aligned}$$

Lemma 9.11 This restricts to addition on Q and to Moore's definition on intervals (Definition 3.4), *i.e.*

$$(\delta_a, \upsilon_s) \ + \ (\delta_b, \upsilon_t) \ = \ (\delta_{a+b}, \ \upsilon_{s+t}).$$

Proof Since Q is itself an ordered Abelian group,

$$\exists d_1 d_2.\, (d < d_1 + d_2) \wedge (d_1 < a) \wedge (d_2 < b) \ \Rightarrow \ d < a + b.$$

Conversely, let $d < d' < d'' < a + b$ by interpolation, $d_1 \equiv a - (a + b - d'') < a$ and $d_2 \equiv b - (d'' - d') < b$, so $d_1 + d_2 = d'$. This proves the equation $\delta_a + \delta_b = \delta_{a+b}$, and that for υ is similar. □

Lemma 9.12 Addition takes cuts to cuts.

Proof $(\delta_1, \upsilon_1) + (\delta_2, \upsilon_2)$ is *rounded* by transitivity and interpolation in $(Q, <)$, keeping the same d_1, d_2, u_1, u_2. It is *bounded* because (δ_1, υ_1) and (δ_2, υ_2) are bounded, and by extrapolation in $(Q, <)$. It is *disjoint* by Lemma 9.8: writing $z(d, u) \equiv \delta d \wedge \upsilon u$ where $z \equiv (\delta, \upsilon)$,

$$(x + y)(d, u) \ \Rightarrow \ \exists qr{:}Q.\ (q + r)(d, u) \ \Leftrightarrow \ \exists qr.\, (d < q + r < u) \ \Rightarrow \ (d < u).$$

For *locatedness*, let $d < u$. By Proposition 9.3, let $d_1, u_1 : Q$ such that $\delta_1 d_1$, $\upsilon_1 u_1$ and $(u_1 - d_1) < (u - d)$, so $d_1 < u_1$ and $d - d_1 < u - u_1$. Then by interpolation there are $d_2, u_2 : Q$ with $d - d_1 < d_2 < u_2 < u - u_1$, so $d < d_1 + d_2$, $u_1 + u_2 < u$ and (by locatedness) $\delta_2 d_2 \vee \upsilon_2 u_2$. □

Proposition 9.13 R is an ordered Abelian group.

Proof $(0, +, -)$ on R satisfy the relevant laws by Lemma 9.9. □

10 Multiplication

Multiplication is more complicated than addition, as we have to characterise $d < a \star b < u$ by a single formula that works independently of the signs of $a, b : R$.

Notation 10.1 For $(\delta, \upsilon) : \Sigma^Q \times \Sigma^Q$ and $q : Q$, let $(\delta, \upsilon) \star q : \Sigma^Q \times \Sigma^Q$ be

$$\begin{pmatrix} \lambda d{:}Q. & (\exists e{:}Q.\ d < eq \wedge \delta e \vee q < 0) \ \wedge \ (\exists t{:}Q.\ d < tq \wedge \upsilon t \vee q > 0) \ , \\ \lambda u{:}Q. & (\exists e{:}Q.\ eq < u \wedge \delta e \vee q > 0) \ \wedge \ (\exists t{:}Q.\ tq < u \wedge \upsilon t \vee q < 0) \end{pmatrix}.$$

Lemma 10.2 If Q is discrete then, for $f, s, q : Q$,

$$(\delta_f, \upsilon_s) \star q \ = \ \begin{cases} (\delta_{fq}, \upsilon_{sq}) & \text{if } q \geq 0 \\ (\delta_{sq}, \upsilon_{fq}) & \text{if } q \leq 0 \end{cases}$$

and in particular $\star$ agrees with multiplication on Q.

Proof We first distinguish the cases in the more general definition according to the sign of $q : Q$. After that we specialise to rational cuts and use roundedness in Axiom 9.1.

$$\begin{array}{rcll} (\delta, \upsilon) \star q & = & (\lambda d.\, \exists e.\, d < eq \wedge \delta e, \ \lambda u.\, \exists t.\, tq < u \wedge \upsilon t) & \text{if } q \geq 0 \\ & = & (\lambda d.\, d < 0, \ \lambda u.\, 0 < u) \ \equiv \ 0 & \text{if } q = 0 \text{ and } (\delta, \upsilon) \text{ bounded} \\ & = & (\lambda d.\, \exists t.\, d < tq \wedge \upsilon t, \ \lambda u.\, \exists e.\, eq < u \wedge \delta e) & \text{if } q \leq 0 \\ (\delta_f, \upsilon_s) \star q & = & \big(\lambda d.\, \exists e.\, (d < eq) \wedge (e < f), \ \lambda u.\, \exists t.\, (s < t) \wedge (tq < u)\big) & \text{if } q \geq 0 \\ & = & (\lambda d.\, d < fq, \ \lambda u.\, sq < u) \ \equiv \ (\delta_{fq}, \upsilon_{sq}) & \\ (\delta_f, \upsilon_s) \star q & = & \big(\lambda d.\, \exists t.\, (d < tq) \wedge (s < t), \ \lambda u.\, \exists e.\, (e < f) \wedge (eq < u)\big) & \text{if } q \leq 0 \\ & = & (\lambda d.\, d < sq, \ \lambda u.\, fq < u) \ \equiv \ (\delta_{sq}, \upsilon_{fq}). & \end{array}$$

□

Lemma 10.3 If Q is discrete then R is an ordered Q-module.

Proof We must show that if (δ, υ) is a cut then so is $(\delta, \upsilon) \star q$, but it is rounded, bounded and disjoint by the same argument as in Lemma 9.12. For locatedness, first suppose that $d < u$ and $q > 0$, so approximate division gives m with $d < mq < u$. Then using roundedness of division and locatedness of (δ, υ), we have

$$\exists et.\, (d < eq) \wedge (e < m < t) \wedge (tq < u) \wedge (\delta e \vee \upsilon t),$$

so $(\exists e.\, d < eq \wedge \delta e) \vee (\exists t.\, tq < u \wedge \upsilon t)$. The case with $q < 0$ is similar, whilst that for $q \equiv 0$ is trivial. The laws for an ordered Q-module follow from Lemma 9.9. □

Remark 10.4 We now know how to define the relations $d < e \star q$ and $t \star q < u$ in the case where q is real but e and t are rational. (Using Lemma 8.3, d and u could also be real.) Since the multiplication formula itself does not involve case analysis, we may substitute these relations into it for the entirely rational ones, an thereby define an operation with two real arguments. But before we can re-apply Lemma 10.3, we have to define approximate division.

Lemma 10.5 R has approximate division with rational quotient, *i.e.*

$$d, q, u : R \;\vdash\; (d < u) \;\wedge\; (q \neq 0) \;\Rightarrow\; \exists m{:}Q.\, (d < m \star q < u).$$

This is ***rounded*** in the sense that, for any $d, q, u : R$ and $m : Q$,

$$(q > 0) \;\wedge\; (d < mq < u) \;\Rightarrow\; \exists et{:}Q.\, (d < e \star q) \wedge (e < m < t) \wedge (q \star t < u).$$

Proof We have four similar cases, with $q > 0$ or $q < 0$ and $u > 0$ or $d < 0$, of which we consider $q, u > 0$. Using $\mathsf{max}(0, d)$ and interpolation, without loss of generality $0 < d < u : Q$. We also put $(\delta, \upsilon) \equiv q$.

Let $0 < p : Q$ with $\delta p \equiv (p < q)$ by roundedness of δ. By approximate division in Q, let $0 < r : Q$ with $u < pr$, let $0 < \ell : Q$ with $2\ell < u - d$ and let $0 < \epsilon : Q$ with $\epsilon r < \ell$. By Proposition 9.3 let $0 < p < e < t$ with $\delta e \wedge \upsilon t \wedge (t - e < \epsilon)$. By approximate division in Q, let $m : Q$ with $d < em < d + \ell$. Then $em < d + \ell < u < pr < er$, so $m < r$ since $e > 0$ and Q is a totally ordered ring. Then $(t - e)m < \epsilon m < \epsilon r < \ell$, so $d < em < tm < em + \ell < d + 2\ell < u$. Hence

$$\exists emt{:}Q.\; 0 < d < em \wedge \delta e \wedge \upsilon t \wedge tm < u.$$

which, by Notation 10.1, says that $d < m \star q < u$. In this case roundedness does follow from Corollary 7.10. □

Proposition 10.6 R is an ordered commutative ring.

Proof We may replace Q by R in Lemma 10.3, except that we need to prove locatedness of $a \star b$ without the case analysis involving $b < 0$, $b \equiv 0$ and $b > 0$.

By commutativity or a symmetrical argument, we have locatedness of $a \star b$ whenever either $a \neq 0$ or $b \neq 0$, so we need only consider a neighbourhood of $(0, 0)$. To be precise, for $a, b, d, u : R$, if $d < u$ then $d < 0 \vee 0 < u$, so $0 < \mathsf{max}(u, -d)$ by Proposition 8.8 and

$$(a > 0) \;\vee\; (a < 0) \;\vee\; (b > 0) \;\vee\; (b < 0) \;\vee\; \big(|a| < 1 \;\wedge\; (|b| < -d \vee |b| < u)\big),$$

in which only the last case remains to be considered. Let $-1 < (\delta, \upsilon) \equiv a < 1$, so $\delta e \wedge \upsilon t \Leftrightarrow \top$, where $e \equiv -1$, $t \equiv +1$ and $q \equiv b$. Then

$$(|b| < -d) \;\equiv\; (d < -|q|) \;\Rightarrow\; (d < -q \vee q < 0) \wedge (d < q \vee q > 0) \;\Rightarrow\; \delta' d$$

and

$$(|b| < u) \;\equiv\; (|q| < u) \;\Rightarrow\; (-q < u \vee q > 0) \wedge (q < u \vee q < 0) \;\Rightarrow\; \upsilon' u,$$

where $a \star b \equiv (\delta, \upsilon) \star q \equiv (\delta', \upsilon')$. Hence $d < u \;\Rightarrow\; \delta' d \vee \upsilon' u$ in all cases. □

Reciprocals and roots are examples of inverses of strictly monotone functions. We set up the general theory in Proposition 8.11, but this must be modified to handle a restricted domain (the positive reals) and order-reversing functions.

Theorem 10.7 R is an ordered field, in which x^{-1} is defined for $x \neq 0$ by

$$\begin{aligned}(\delta, \upsilon)^{-1} \quad &\equiv \quad \big(\lambda d.\, \exists u.\, \upsilon u \land ((du < 1 \land \delta 0) \lor (1 < du \land d < 0)),\\ &\qquad \lambda u.\, \exists d.\, \delta d \land ((du < 1 \land u > 0) \lor (1 < du \land \upsilon 0))\big).\end{aligned}$$

In the strictly positive or negative cases this is respectively

$$\big(\lambda d.\, (d < 0) \lor (\exists u.\, \upsilon u \land du < 1),\ \lambda u.\, (u > 0) \land (\exists d.\, \delta d \land du < 1)\big)$$

or

$$\big(\lambda d.\, (d < 0) \land (\exists u.\, \upsilon u \land 1 < du),\ \lambda u.\, (u > 0) \lor (\exists d.\, \delta d \land 1 < du)\big).$$

Proof In the positive case, we claim that the relations $d \lessdot u$ and $u \ltimes u$ given by $(du < 1)$ and $(1 < du)$ define a contravariant strictly monotone graph.

The first four axioms of the form $0 < ax < 1 \;\Rightarrow\; \exists b.\, ax < bx < 1$, by approximate division. For the other two, given $0 < a < b$, by approximate division let $r, x : Q$ such that $a < rb$ and $r < ax < 1$; then $a < br < bax$, so $1 < bx$.

The negative case is similar, and we observe that the given formula combines the two cases. Of course we do not need to consider 0 this time. □

Remark 10.8 Again this agrees with Moore's formulae (Definition 3.4) in the legitimate case $0 < a \leq b$, where

$$(\delta_a, \upsilon_b)^{-1} \;=\; (\delta_{1/b}, \upsilon_{1/a}) \quad \text{and} \quad (\delta_{-b}, \upsilon_{-a})^{-1} \;=\; (\delta_{-1/a}, \upsilon_{-1/b}),$$

assuming for convenience here that Q is a field. Since

$$\exists du.\, (\delta, \upsilon)^{-1}(d, u) \;\Rightarrow\; \delta 0 \lor \upsilon 0,$$

the value in any illegitimate case, including 0^{-1} and $(\delta_{-1}, \upsilon_1)^{-1}$, is $(\bot, \bot)$, which denotes the interval $[-\infty, +\infty]$. □

Proposition 10.9 R has $(2n+1)$st roots and $[0, \infty)$ has $2n$th roots, where

$$\begin{aligned}\sqrt[2n+1]{(\delta, \upsilon)} \quad &\equiv \quad \big(\lambda d.\, \delta(d^{2n+1}),\ \lambda u.\, \upsilon(u^{2n+1})\big)\\ +\sqrt[2n]{(\delta, \upsilon)} \quad &\equiv \quad \big(\lambda d.\, (d < 0) \lor \delta(d^{2n}),\ \lambda u.\, (u > 0) \land \upsilon(u^{2n})\big).\end{aligned}$$

In the illegitimate case, $\sqrt{-1} = 0$.

Proof The strictly monotone graph is given by the relations $(d < e^n)$ and $(t^n < u)$. Five of the interpolation properties are proved by m-fold ($m \equiv 2n$ or $2n+1$) applications of roundedness of division: for $c < e^m$,

$$\exists d_1 d_2 \ldots d_m.\, c < d_1 d_2 \cdots d_m < \cdots < d_1 d_2 e^{m-2} < d_1 e^{m-1} < e^m \;\land\; \mathsf{max}(d_1, \ldots, d_m) < e,$$

so $c < d^m$, where $d \equiv \mathsf{max}(d_1, \ldots, d_m) < e$. For the last we need approximate roots in Q:

$$d, u : Q \;\vdash\; 0 < d < u \;\Rightarrow\; \exists x{:}Q.\, d < x^n < u.$$

These may be found in an Archimedean ordered commutative ring with approximate division by methods for which we cite Babylonian clay tablets as the original source. □

11 The closed interval

Now we turn to the Heine–Borel Theorem, that the closed interval is compact, in the sense that it admits a "universal quantifier" $\forall$. But first, of course, we have to construct the closed interval itself, and for the sake of symmetry the open one too. These are defined by means of terms of type Σ^R.

Definition 11.1 For $d \leq u : R$, the ***open*** and ***closed intervals*** $(d, u), [d, u] \subset R$ are
(a) the open subspace classified by $(\lambda x.\, d < x < u) \equiv (\upsilon_d \wedge \delta_u) : \Sigma^R$, and
(b) the closed subspace co-classified by $(\lambda x.\, x < d \vee u < x) \equiv (\delta_d \vee \upsilon_u) : \Sigma^R$ respectively.

Next we have to construct the ASD subobjects that these terms (co)classify. Since we have not allowed dependent types in the calculus[2], we must temporarily restrict attention to intervals with constant endpoints, so d and u cannot have parameters. In other words, we are really just dealing with the *unit* interval $\mathbb{I} \equiv [0, 1]$, and $d \equiv 0$, $u \equiv 1$. But this is no real handicap, since the general interval $[d, u]$ is the direct image of $[0, 1]$ under the function $t \mapsto d(1 - t) + ut$. The outcome of this is that the formulae that we obtain in the end remain valid with parametric endpoints.

Remark 11.2 Let $\psi : \Sigma^R$ be a closed term (*i.e.* without parameters), which we think of as a continuous function $\psi : R \to \Sigma$. The open and closed subspaces that ψ (co)classifies are the inverse images $U \equiv \phi^{-1}(\top)$ and $C \equiv \phi^{-1}(\bot)$ of the two points of the Sierpiński space. We have to construct them using nuclei (Section 5), but fortunately this is easier than it was for R itself.

Classically, any open subspace of U is already one of R, whilst an open subspace of C becomes open in R when we add U to it. This means that

$$\Sigma^U \;\cong\; \Sigma^R \downarrow \psi \;\equiv\; \{\phi \mid \phi \leq \psi\} \quad \text{and} \quad \Sigma^C \;\cong\; \psi \downarrow \Sigma^R \;\equiv\; \{\phi \mid \phi \geq \psi\}.$$

From this we see what the Σ-splittings and nuclei for U and C must be.

Lemma 11.3 The idempotents $(-)\wedge\psi$ and $(-)\vee\psi$ are nuclei on Σ^R (Definition 5.6), with respect to which $a : R$ is admissible (Definition 5.7) iff $\psi a \Leftrightarrow \top$ or $\psi a \Leftrightarrow \bot$, respectively.

Proof The equations in Lemma 5.5 follow from the distributive law. However, the Phoa principle (Axiom 4.9) is needed to show that they satisfy the abstract λ-equation in [B] that is more directly related to the monadic principle [C]. $\square$

Notation 11.4 As we shall need to name the Σ-splittings of $j : U, C \rightarrowtail R$, we write $\exists_j : \Sigma^U \rightarrowtail \Sigma^R$ and $\forall_j : \Sigma^C \rightarrowtail \Sigma^R$, with $\exists_j(\Sigma^j\phi) = \phi \wedge \psi$ and $\forall_j(\Sigma^j\phi) = \phi \vee \psi$. See [C] for the reasons for these names, in terms of which the following result says that $\exists_{(d,u)} \equiv \exists_R \cdot \exists_j$.

Proposition 11.5 Any open subspace $U \subset R$ classified by $\psi : \Sigma^R$, in particular the open intervals (d, u) and (υ, δ), are overt, with, for $\phi : \Sigma^R$,

$$\begin{aligned} \exists x{:}U.\, \phi x \quad &\equiv \quad \exists x{:}R.\, \psi x \wedge \phi x \\ \exists x{:}(d, u).\, \phi x \quad &\equiv \quad \exists x{:}R.\, (d < x < u) \wedge \phi x \end{aligned}$$

where $\exists x : R$ was defined in Proposition 8.1.

[2] We argue in [J] that it is better to work with (co)classifiers and modal operators than with the (sub)spaces themselves and their quantifiers. Besides avoiding the introduction of dependent types, *terms* with parameters have been familiar for centuries, and translate more directly into programs. However, if we tried to prove the results in this section in that way, we would still have to demonstrate that the formulae that *look* like nuclei or modal operators *actually* correspond to (dependent) subspaces, which is at least as difficult as the approach that we take here.

Proof We may deduce in both directions:

$$\begin{array}{llll} \Gamma,\ x:U & \vdash & \phi x \ \Rightarrow\ \sigma & \\ \Gamma,\ x:R,\ \psi x \Leftrightarrow \top & \vdash & \phi x \ \Rightarrow\ \sigma & \\ \Gamma,\ x:R & \vdash & \psi x\ \wedge\ \phi x\ \Rightarrow\ \sigma & \text{Lemma 4.11} \\ \Gamma & \vdash & (\exists x{:}R.\ \psi x \wedge \phi x)\ \Rightarrow\ \sigma & R \text{ overt} \end{array}$$

so $\exists x{:}R.\ \psi x \wedge \phi x$ satisfies the defining property of $\exists x{:}U.\ \phi x$. □

Turning to the main question of the compactness and overtness of the *closed* interval, we follow a strategy that was suggested by the CCA referee.

Lemma 11.6 $Y \equiv \Sigma^Q \times \Sigma^Q$ is overt compact, with $\exists_Y \Phi \equiv \Phi(\top, \top)$ and $\forall_Y \Phi \equiv \Phi(\bot, \bot)$. □

Lemma 11.7 Let $i : X \rightarrowtail Y$ and $I : \Sigma^X \rightarrowtail \Sigma^Y$ with $\Sigma^i \cdot I = \mathsf{id}$. Then
(a) if I preserves $\bot$ and Y is overt then so is X, with $\exists x{:}X.\ \phi x \ \equiv\ \exists y{:}Y.\ I\phi y$, whilst
(b) if I preserves $\top$ and Y is compact then so is X, with $\forall x{:}X.\ \phi x \ \equiv\ \forall y{:}Y.\ I\phi y$.

Proof In each column we deduce in both directions,

$$\begin{array}{lll} \Gamma,\ x:X & \vdash & \phi x\ \Rightarrow\ \sigma \\ \Gamma,\ \sigma \Leftrightarrow \bot & \vdash & \Sigma^i(I\phi)\ \equiv\ \phi\ =\ \bot : \Sigma^X \\ \Gamma,\ \sigma \Leftrightarrow \bot & \vdash & I\phi\ =\ \bot : \Sigma^Y \\ \Gamma,\ \sigma \Leftrightarrow \bot & \vdash & \exists_Y(I\phi)\ \Rightarrow\ \bot \\ \Gamma & \vdash & \exists_Y(I\phi)\ \Rightarrow\ \sigma \end{array} \qquad \begin{array}{lll} \Gamma,\ x:X & \vdash & \sigma\ \Rightarrow\ \phi x \\ \Gamma,\ \sigma \Leftrightarrow \top & \vdash & \top\ =\ \phi\ \equiv\ \Sigma^i(I\phi) : \Sigma^X \\ \Gamma,\ \sigma \Leftrightarrow \top & \vdash & \top\ =\ I\phi : \Sigma^Y \\ \Gamma,\ \sigma \Leftrightarrow \top & \vdash & \top\ \Rightarrow\ \forall_Y(I\phi) \\ \Gamma & \vdash & \sigma\ \Rightarrow\ \forall_Y(I\phi) \end{array}$$

using Lemma 4.11 and Definition 4.14. □

Proposition 8.1 has already used this argument to show that R is overt, using the standard Σ-splitting $I : \Sigma^R \rightarrowtail \Sigma^{\Sigma^Q \times \Sigma^Q}$, which preserves $\bot$. However, the Σ-splitting $I \cdot \forall_j$ that we used above to construct the closed interval $[d, u]$ preserves neither $\top$ or $\bot$, so we have to devise new ones that do.

Notation 11.8 Let $\theta : \Sigma^{[d,u]}$ be the restriction of $\phi : \Sigma^R$ and $\Phi : \Sigma^{\Sigma^Q \times \Sigma^Q}$, for which the canonical choices are $\phi \equiv \forall_j \theta$ and $\Phi \equiv I(\forall_j \theta)$. Then define

$$\begin{array}{lll} I_{\langle d,u\rangle}\theta(\delta, \upsilon) & \equiv & I(\forall_j\theta)(\delta \wedge \delta_u, \upsilon \wedge \upsilon_d) \\ I_{[d,u]}\theta(\delta, \upsilon) & \equiv & I(\forall_j\theta)(\delta \vee \delta_d, \upsilon \vee \upsilon_u) \end{array} \qquad \begin{array}{lll} \mathcal{E}_{\langle d,u\rangle}\Phi(\delta, \upsilon) & \equiv & \mathcal{E}\Phi(\delta \wedge \delta_u, \upsilon \wedge \upsilon_d) \\ \mathcal{E}_{[d,u]}\Phi(\delta, \upsilon) & \equiv & \mathcal{E}\Phi(\delta \vee \delta_d, \upsilon \vee \upsilon_u). \end{array}$$

Lemma 11.9 $I_{\langle d,u\rangle}$ and $I_{[d,u]}$ are both Σ-splittings of $[d, u] \rightarrowtail \Sigma^Q \times \Sigma^Q$, with nuclei $\mathcal{E}_{\langle d,u\rangle}$ and $\mathcal{E}_{[d,u]}$ respectively.

Proof For $d \le x \le u$, Proposition 6.13 says that $\delta_d \le \delta_x \le \delta_u$ and $\upsilon_d \ge \upsilon_x \ge \upsilon_u$. Then, since $I \cdot \forall_j$ is already a Σ-splitting,

$$\begin{array}{lllllll} I_{\langle d,u\rangle}\theta(\delta_x, \upsilon_x) & \equiv & I(\forall_j\theta)(\delta_x \wedge \delta_u, \upsilon_x \wedge \upsilon_d) & \Leftrightarrow & I(\forall_j\theta)(\delta_x, \upsilon_x) & \Leftrightarrow & \theta x \\ I_{[d,u]}\theta(\delta_x, \upsilon_x) & \equiv & I(\forall_j\theta)(\delta_x \vee \delta_d, \upsilon_x \vee \upsilon_u) & \Leftrightarrow & I(\forall_j\theta)(\delta_x, \upsilon_x) & \Leftrightarrow & \theta x. \end{array}$$ □

Lemma 11.10 $I_{\langle d,u\rangle}$ and $I_{[d,u]}$ preserve $\bot$ and $\top$ respectively.

Proof Since $\theta \equiv \top, \bot$ are the restrictions of $\Phi \equiv \top, \bot$,

$$\begin{array}{lllll} I_{\langle d,u\rangle}\bot(\delta, \upsilon) & \Leftrightarrow & \mathcal{E}\bot(\delta \wedge \delta_u, \upsilon \wedge \upsilon_d) & \Leftrightarrow & \bot \\ I_{[d,u]}\top(\delta, \upsilon) & \Leftrightarrow & \mathcal{E}\top(\delta \vee \delta_d, \upsilon \vee \upsilon_u) & \Leftarrow & \mathcal{E}\top(\delta_d, \upsilon_u) \\ & \Leftrightarrow & \exists q < r.\ \delta_d q \wedge \upsilon_u r & \Leftrightarrow & \top \end{array}$$

by inspection of Notation 7.3, in which $\mathcal{E}$ has at least one conjunct. Notice that the last step says that $[d, u] \subset (q, r)$, *i.e.* it is a *bounded* closed subspace of R. □

Theorem 11.11 For $d \leq u : R$, the closed interval $[d, u]$ is overt and compact, with

$$\begin{array}{lllllll} \exists x{:}[d,u].\ \phi x & \equiv & \exists x{:}[d,u].\ I\phi(\delta_x, \upsilon_x) & \equiv & I_{\langle d,u\rangle}\phi(\top,\top) & \Leftrightarrow & I\phi(\delta_u, \upsilon_d) \\ \forall x{:}[d,u].\ \phi x & \equiv & \forall x{:}[d,u].\ I\phi(\delta_x, \upsilon_x) & \equiv & I_{[d,u]}\phi(\bot,\bot) & \Leftrightarrow & I\phi(\delta_d, \upsilon_u) \end{array} \qquad \square$$

Remark 11.12 You are probably wondering where the "finite open sub-cover" has gone. Essentially, it was absorbed into the axioms, in the form of general Scott continuity (Axiom 4.20).

Compactness is often said to be a generalisation of finiteness. We dissent from this. The finiteness is a side-effect of Scott continuity. The essence of *compactness* lies, not in that the cover be finite, but in what it says about the notion of *covering*. Recall how the *statement* of equality became an (in)equality *predicate* in a discrete or Hausdorff space (Remark 4.7). Similarly, compactness promotes the judgement that an open subspace covers into a predicate,

$$[d,u]\phi \ \equiv\ \forall x{:}[d,u].\ \phi x \ \Leftrightarrow\ I\phi(\delta_d, \upsilon_u).$$

Now, $\lambda\phi.\ [d,u]\phi$ is a term in the ASD λ-calculus of type Σ^{Σ^R}, and, like all such terms, preserves directed joins, so

$$\text{if } \Gamma,\ n : N,\ x : R\ \vdash\ \phi_n x \text{ is directed} \quad \text{then} \quad \Gamma\ \vdash\ [d,u] \bigvee_n^{\uparrow} \phi_n\ \Leftrightarrow\ \bigvee_n^{\uparrow} [d,u]\phi_n.$$

Restating this for general joins, we have

$$[d,u]\exists n{:}N.\ \phi_n\ \Leftrightarrow\ \exists \ell{:}\mathsf{Fin}(N).\ [d,u]\exists n \in \ell.\ \phi_n$$

where ℓ is the finite open sub-cover. $\square$

Warning 11.13 Our "$\exists \ell : \mathsf{Fin}(N)$" does not have the same strength as it does in other constructive systems. The computational implementation only *produces* the finite set ℓ of indices in the case where the predicate (is provably true and) has at worst variables of type $\mathbb{N}$ (Remark 2.5). In particular, if the predicate ϕ (and maybe even the bounds d and u) are expressions with real-valued parameters, the list ℓ (and its length) cannot be extracted as functions of these parameters.

Warning 11.14 In view of the fact that both quantifiers are given by almost the same formula, the hypothesis $d \leq u$ must be important. If we don't know this, we can still obtain the bounded *universal* quantifier as

$$\phi : \Sigma^R,\ d, u : R\ \vdash\ \big(\forall d \leq x \leq u.\ \phi x\big)\ \equiv\ (d > u)\ \vee\ I\phi(\delta_d, \upsilon_u),$$

but for the *existential* quantifier we must write

$$\phi : \Sigma^R,\ d \leq u : R\ \vdash\ \exists d \leq x \leq u.\ \phi x\ \equiv\ I\phi(\delta_u, \upsilon_d).$$

This is because $(\exists d \leq x \leq u.\ \top) \Leftrightarrow \top\ \dashv\vdash\ (d \leq u)\ \dashv\vdash\ (d > u) \Leftrightarrow \bot$, making a positive statement equivalent to a negative one. But that would make the predicate $(d > u)$ decidable, which it can only be if we had already *assumed* that it was either true or (in fact) false.

Notice that the intersection of two overt subspaces (such as two closed intervals) need not be overt, since we need to be able to decide whether they overlap or not. We shall see more situations of this kind in [J]. $\square$

Remark 11.15 While we're looking at awkward cases, recall from Corollary 8.1 that

$$\phi : \Sigma^R\ \vdash\ \exists x{:}R.\ \phi x\ \Leftrightarrow\ I\phi(\top,\top),$$

and it's easy to check that $\exists x \leq u.\ \phi x\ \Leftrightarrow\ I\phi(\delta_u, \top)$ and $\exists x \geq d.\ \phi x\ \Leftrightarrow\ I\phi(\top, \upsilon_d)$.

So what happens if we let one or both of the arguments of $I\phi$ be $\bot$? Does this give a meaning to $\phi(\infty) \equiv \mathsf{lim}_{x\to\infty}\phi x$ or to $\forall -\infty \leq x \leq \infty.\ \phi x$? Unfortunately not: putting δ or $\upsilon \equiv \bot$ in Notation 7.3 just gives $\bot$. $\square$

12 Recursive analysis

Whether the closed interval is, or ought to be, compact in computable and constructive mathematics is a question that never ceases to be interesting. This is so partly because compactness is such an important property for a space to have, and partly because there are several schools of constructive and computable mathematics that disagree about the answer, each of which sets up its own framework, differing slightly from all the others. The matter is made all the more confusing by the number of different definitions of compactness that have been given [Bou66]. In this concluding section we make a few observations about compactness of the closed interval in ASD and how it relates to various other schools of computability.

We hear alarm bells ringing in the minds of recursive analysts. On the one hand, in Remark 2.4 we saw that there is a model of ASD in which maps are precisely the computable maps between computably based locally compact spaces. On the other, Section 11 proved that the closed interval is compact. If all functions are computable, then surely we should be able to use some enumeration of the definable points of the closed interval to show that it is *not* compact?

Remark 12.1 In Recursive Analysis, the closed interval $[d, u]$ can be shown *not* to be compact by means of a ***singular cover***, *i.e.* a countable cover by open intervals with rational endpoints whose total length is bounded above by some $\epsilon < d - u$. It can be shown that no finite subcover of a singular cover is a cover of $[d, u]$, see [TvD88, §6.4.3] or [BR87, §3.4]. Such covers exist in Recursive Analysis [TvD88, §6.4.2] because of the *formal Church's Thesis* CT_0 [TvD88, §3.3.2], which states that every function $\mathbb{N} \to \mathbb{N}$ has a Gödel code. It follows that every real number has a Gödel code, too. In classical recursion theory a singular cover is described as a computable enumeration of open intervals with rational endpoints of bounded total length which effectively covers all computable reals.

Example 12.2 The closed interval may also fail to be compact for reasons that have nothing to do with computability. The construction of the "Dedekind reals" R *via* cuts as we did it in Section 6 can be done in any category with finite limits, certain exponentials and some mild assumptions on $\Sigma^{\mathbb{N}}$, as in Section 4. An example is the category **Dcpo** of posets with directed joins and Scott-continuous functions. Here the equaliser in Remark 7.1 has the discrete ("specialisation") order, but in this category the order determines the topology. The object R therefore also carries the discrete topology, so Σ^R is the powerset $\mathcal{P}(R)$, and the only compact subsets of R are the finite ones. The formula in Proposition 3.7 gives an element of $\Sigma^{\Sigma^{\mathbb{Q}} \times \Sigma^{\mathbb{Q}}}$ for each $U \subset \mathbb{R}$, but the function I is not Scott-continuous, so does not yield a sound interpretation of the ASD calculus (Remark 5.10).

Remark 12.3 Another approach is that of *relative computability*, in which all maps are computable, but the spaces are the classical topological ones, *i.e.* they have "all" points instead of just the computable ones. Each space is *also* equipped with a "computability" structure, such as an enumeration of basic open sets, that is used to specify which continuous maps are computable. Examples are (one variant of) Type Two Effectivity [Wei00], and computable equilogical spaces [Bau00]. Another is Martín Escardó's synthetic topology of data types [Esc04]; whilst this is very similar to the calculus in Section 4, it lacks the crucial monadic property of ASD (Section 5). In all of these three settings, the closed interval is compact, the proof being the well known classical one. In particular, Escardó stresses the need for "all" points, as compactness would fail in a purely recursive version of his theory.

Remark 12.4 The skeptical pupil of the Russian School will not be so easily convinced, but will press us on the structure of the *syntactic* term model of ASD, in which the objects are types and the morphisms are the definable maps. Since everything is enumerable in such a model, one might expect to obtain a singular cover, and so non-compactness of the closed interval. Indeed, by following the usual construction, we could define a sequence of intervals with rational endpoints,

(a_n, b_n), whose total length is bounded by $\epsilon < 1$, and prove the meta-theorem that

$$\text{“if} \quad \vdash t : [0,1] \quad \text{then} \quad \vdash \exists n{:}\mathbb{N}.\ a_n < t < b_n\text{”}.$$

However, it does not follow that (a_n, b_n) *covers* $[0, 1]$, by which we mean

$$x : [0,1] \ \vdash\ \exists n{:}\mathbb{N}.\ a_n < x < b_n,$$

because that would be to confuse a *family* of theorems about *all definable terms* t with a *single* theorem containing a free variable x. In category theory, the calculus is said to be *not well pointed.* Even though its morphisms are recursively enumerable, there are not enough of them $\mathbf{1} \to R$ to cover the equaliser.

Example 12.5 As a final attempt to break compactness of the closed interval in ASD, we might try to interpret it in a setting, such as Markov's Recursive Mathematics, in which the formal Church's Thesis CT_0 holds. A useful formulation of this is the category **PER** of partial equivalence relations over Stephen Kleene's first algebra, or Martin Hyland's larger *effective topos*, **Eff** [Hyl82].

We try to interpret ASD in **PER** or **Eff** in the natural way, using the universal properties of $\mathbb{N}$, products, equalisers and exponentials. For Σ we take the object of semidecidable propositions. This yields a valid interpretation of the language and construction of Sections 4 and 6, so that the equaliser R in Remark 7.1 is perhaps a reasonable candidate for the “real line” in these worlds.

However, we do not have a sound interpretation of Σ-split subspaces, because the map I that Remark 5.10 requires need not exist. In particular, the object Σ^R does not have the properties of the “Euclidean” topology, and the closed interval is not compact.

In **PER**, the formula in Proposition 3.7 does not define a morphism I because $[d, u] \subset U$ is not a completely r.e. predicate (in d, u and U). In the internal language of **Eff**, the sub-expression $\forall x{:}[d, u].\ x \in U$ is a term of type Ω but not Σ. Consequently, the formula defines a morphism I of type $\Omega^{\Sigma^Q \times \Sigma^Q}$ instead of $\Sigma^{\Sigma^{\mathbb{Q}} \times \Sigma^{\mathbb{Q}}}$.

This situation is better documented for Cantor space:

Example 12.6 Richard Friedberg [Fri58] [RJ92, §15.3.XXXI] defined an effective operation on the set of total recursive functions $\mathbb{N} \to \mathbb{N}$ that is not the restriction of a recursive functional.

In the effective topos, $\mathbf{2}^{\mathbb{N}}$ and $\mathbb{N}^{\mathbb{N}}$ are actually isomorphic (so they cannot be locally compact). We can reformulate Friedberg's example by saying that the topology of $\mathbf{2}^{\mathbb{N}}$ (*i.e.* the object $\Sigma^{\mathbf{2}^{\mathbb{N}}}$) is not the subspace topology induced by the inclusion $\mathbf{2}^{\mathbb{N}} \subset \mathbf{2}_\perp^{\mathbb{N}}$. On the other hand, $\mathbf{2}^{\mathbb{N}}$ can be defined in ASD by means of a nucleus on $\mathbf{2}_\perp^{\mathbb{N}}$, so it has the subspace topology and is locally compact, indeed compact Hausdorff.

Remark 12.7 You may be left thinking that models of ASD are complicated, mysterious and hard to find. In fact, the calculus has interpretations in many kinds of classical or constructive set, type or topos theory. It is more accurate to say that these interpretations can be *based on* other foundational systems, because we obtain a model of our calculus by doing a *construction* on top of a system with weaker properties. In other words, whilst the direct interpretation in **PER** and **Eff** in Example 12.5 does not work, a slightly more involved one does.

Theorem 12.8 Let $(\mathcal{C}, \Sigma)$ be a model of the axioms in Section 4, in particular with products and powers Σ^X. Suppose that idempotents split in $\mathcal{C}$. Then $\mathcal{A}^{\mathsf{op}}$ is a model of ASD, where $\mathcal{A}$ is the category of Eilenberg–Moore algebras for the monad arising from $\Sigma^{(-)} \dashv \Sigma^{(-)}$ [B]. □

Remark 12.9 Any category that results from this construction contains (the sober objects of) the original one, and more, and the embedding preserves $\Sigma^{(-)}$. The point is that many other constructions are *not* preserved, important examples being of course Cantor space and the Dedekind reals as we have defined them.

Whilst these objects may have had undesirable properties in the original structure, their analogues in the new one turn out to behave as the mainstream mathematician would expect. The new objects are by definition the exponentials, equalisers, *etc.* for the same data in the *new* category, whilst (the images of) the old objects are relieved of their former duties.

This construction can be carried out in all of the well known models of computable mathematics, including Recursive Mathematics, domain-theoretic **PER** models and Type Two Effectivity. For example, when we apply it to **Dcpo**, we obtain the category **LKLoc** of locally compact locales. It would be interesting and fruitful to do this in detail for the other cases, and in particular to compare their collections of overt objects.

On the other hand, these models have other significant properties besides those in Section 4, which it may not be easy to reproduce in the new category. For example, **Dcpo** is cartesian closed, but **LKLoc** is not.

Theorem 12.10 Over the effective topos **Eff** there is a sheaf topos $\mathbf{Eff}^{\mathsf{A}}$ that has a model $\mathcal{A}^{\mathsf{op}}$ of ASD as a reflective subcategory. (This is unpublished joint work of Giuseppe Rosolini and Paul Taylor).

Proof There is an *internal* category $\mathsf{C} \in \mathbf{Eff}$ of PERs that is weakly equivalent to a *reflective* subcategory $\mathcal{C} \subset \mathbf{Eff}$ of $\Omega_{\neg\neg}$-discrete objects [HRR90]. (This situation is a peculiarity of realisability toposes: Peter Freyd observed in the 1960s that, classically, any small complete category is a lattice [Tay99, Example 7.3.2(k)], and this extends to Grothendieck toposes.)

Now let A be the Eilenberg–Moore category for the monad over C, and $\mathcal{A}$ that over $\mathcal{C}$ (or equivalently over **Eff**). Then by Theorem 12.8, both A^{op} and $\mathcal{A}^{\mathsf{op}}$ are models of ASD.

Since A^{op} is an internal category, *i.e.* a *small* one in classical language, we may form the sheaf topos $\mathbf{Eff}^{\mathsf{A}}$. But A^{op} and $\mathcal{A}^{\mathsf{op}}$ are weakly equivalent to each other, and therefore to a full subcategory of $\mathbf{Eff}^{\mathsf{A}}$ by the Yoneda embedding. On the other hand, $\mathcal{A}^{\mathsf{op}}$ is also *totally cocomplete* (see [Kel86, Corollary 6.5] in particular, but the theory was developed in [SW78, Str80, Tho80, Woo82]), so this full subcategory is *reflective* in $\mathbf{Eff}^{\mathsf{A}}$. □

Remark 12.11 We conclude with some thoughts on the relationship between ASD and the constructivism that Errett Bishop advocated [Bis67]. He was not only constructive but also *conservative*, in the sense that his theorems are compatible both with classical analysis and with Brouwer's Intuitionism and Markov's Recursive Mathematics. Since these settings disagree about Heine–Borel compactness of the closed interval, Bishop focused on Cauchy completeness and total boundedness instead.

Given that ASD proves compactness of the closed interval, his followers may jump to the conclusion that our theory is unacceptable in their system. Before they do so, they should first remember that the spaces in ASD are *not* sets and that the ASD calculus is *not* the calculus of logical predicates about points. The ASD statement "$[0,1]$ is compact" says nothing about the closed interval $[0,1]$ from Bishop's mathematics, because the ASD type R cannot be interpreted naïvely as Bishop's set of real numbers.

One way of understanding the logical strength of ASD is as an algebraic formulation of topology, *cf.* Remark 5.13. When the "ASD algebras" are constructed in Bishop's world, one of them will be called $[0,1]$, and there will be an "ASD homomorphism" $\forall_{[0,1]} : \Sigma^{[0,1]} \to \Sigma$ witnessing the fact that $[0,1]$ behaves like a compact space within the realm of ASD algebras. The interesting question from the point of view of a Bishop-style constructivist is not whether ASD algebras are acceptable *per se* (which they are, so far as we can tell), but how good a formulation of topology ASD provides.

Remark 12.12 On the other hand, ASD is not presented as algebra but as topology. This has been done by taking advantage of forty years' study of categorical logic, type theory and the relationship between the two. Using this, a terse piece of category theory has been transformed into a symbolic calculus that can be used by real analysts. If, despite our advocacy of the intrinsic nature of the topology of the real line, you still believe that sets of points are fundamental, then you will inevitably regard these "algebraic spaces" as artificial. However, this point of view is unsustainable in computation, where *everything* is a mirage — our calculus has at least as good a claim to represent real analysis as does one that is based on Gödel numberings of points.

A final point that might interest Bishop-style constructivists is our basic type Σ. Bishop — and he is certainly not the only one to do so — took *numbers* as fundamental, and proceeded to construct everything else from them. This is supposedly because integers are the basic objects of computation. But modern computer science (in particular domain theory) teaches us that *observable properties* are equally, if not more, fundamental. The space Σ *is* the space of observable propositions. How would Bishop have axiomatised Σ, and more generally spaces of observable properties Σ^X? Perhaps with something fairly close to Abstract Stone Duality.

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