Two Constructive Embedding-Extension Theorems with Applications to Continuity Principles and to Banach-Mazur Computability

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We prove two embedding and extension theorems in the context of the constructive theory of metric spaces. The first states that Cantor space embeds in any inhabited complete separable metric space (CSM) without isolated points, X, in such a way that every sequentially continuous function from Cantor space to $\mathbb{Z}$ extends to a sequentially continuous function from X to $\mathbb{R}$. The second asserts an analogous property for Baire space relative to any inhabited locally non-compact CSM. Both results rely on having careful constructive formulations of the concepts involved.

As a first application, we derive new relationships between "continuity principles" asserting that all functions between specified metric spaces are pointwise continuous. In particular, we give conditions that imply the failure of the continuity principle "all functions from X to $\mathbb{R}$ are continuous", when X is an inhabited CSM without isolated points, and when X is an inhabited locally non-compact CSM. One situation in which the latter case applies is in models based on "domain realizability", in which the failure of the continuity principle for any inhabited locally non-compact CSM, X, generalizes a result previously obtained by Escardó and Streicher in the special case $X = \mathcal{C}[0, 1]$.

As a second application, we show that, when the notion of inhabited complete separable metric space without isolated points is interpreted in a recursion-theoretic setting, then, for any such space X, there exists a Banach-Mazur computable function from X to the computable real numbers that is not Markov computable. This generalizes a result obtained by Hertling in the special case that X is the space of computable real numbers.

1 Introduction

In computable and constructive analysis, it sometimes happens that pathological properties of Baire space are reflected by similar pathologies holding for "continuous" spaces of the sort that arise in analysis. Two examples of such phenomena have appeared in the recent literature.

The first occurs in the context of so-called *domain realizability*, i.e. in realizability toposes constructed over domain-theoretic models of the untyped λ-calculus. In many such models, the internal *continuity principle* "all functions from Baire space to $\mathbb{N}$ are continuous" is known to be false, even though externally all morphisms from Baire space to $\mathbb{N}$ *are* continuous, because it conflicts with choice principles valid in the models. Recently, Escardó and Streicher showed that similarly the internal statement "all functions from $\mathcal{C}[0, 1]$ to $\mathbb{R}$ are continuous" is false [7]. Once again, externally, all morphisms from $\mathcal{C}[0, 1]$ to $\mathbb{R}$ are continuous.

The second example arises in the context of differentiating between computability in the sense of Markov and computability in the sense of Banach and Mazur. It is an old result of Friedberg [8] that there exists a Banach-Mazur-computable function, mapping computable sequences of natural numbers (the computable version of Baire

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space) to $\mathbb{N}$, that is not Markov computable (whereas every Markov computable function is trivially Banach-Mazur computable). Recently, Hertling answered a longstanding open question by proving that, similarly, there exists a Banach-Mazur-computable function on the computable real numbers that is not Markov computable [10].

In neither example above [7, 10] is the pathological behaviour in the analytic world derived from the analogous result for Baire space. Instead, direct proofs are given, borrowing ideas from the known proofs for Baire space, but adapting them, with added complexity, to apply to the spaces under consideration. In this paper, we provide instead a method of deriving such results directly as consequences of the corresponding results for Baire space, and we show that the results of [7, 10] both follow from applications of our method.

Working within the context of Bishop's Constructive Mathematics [5], we identify two properties of complete separable metric spaces (CSMs), namely being *without isolated points* and *local non-compactness*. Despite the negative terminology, as befits the constructive setting, these concepts are formulated as positive properties of spaces. Our first main result, Theorem 2.4, states that Cantor space, which is itself without isolated points, embeds in any inhabited CSM without isolated points, X, in such a way that every sequentially continuous function from Cantor space to $\mathbb{Z}$ extends to a function from X to $\mathbb{R}$. A second result, Theorem 2.5, gives an analogous property for Baire space relative to any inhabited locally non-compact CSM. These results are proved in Section 3.

In Section 4, we apply Theorem 2.5 to derive Escardó and Streicher's result that the continuity principle "all functions from $\mathcal{C}[0,1]$ to $\mathbb{R}$ are continuous" is false in domain realizability [7]. This is a simple consequence of the known result for Baire space, together with the fact that $\mathcal{C}[0,1]$ is easily shown to be locally non-compact. Furthermore, our approach establishes a more general result that, for any inhabited locally non-compact X, the statement "all functions from X to $\mathbb{R}$ are continuous" is false in any constructive setting in which certain choice and sequential-continuity principles are valid. Realizability toposes built from domain-theoretic models of λ-calculus validate these principles. More generally, we establish various relationships between different continuity principles, including the failure of continuity principles involving inhabited CSMs without isolated points, in certain situations in which Theorem 2.4 applies.

In Section 5, we derive Hertling's result, [10], that there exists a Banach-Mazur computable function on the computable real numbers that is not Markov computable, as a consequence of Friedberg's result [8]. More generally, we show that, for any inhabited CSM X without isolated points (understood recursion-theoretically), there exists a Banach-Mazur computable function from X to the computable reals that is not Markov computable.

We believe that Theorems 2.4 and 2.5 may have other applications in computable and constructive analysis, as they appear to provide general tools for extending properties of the Cantor and Baire spaces to other spaces.

2 Two constructive embedding-extension theorems

Following Bishop [4, 5], we do mathematics using intuitionistic logic, and we assume the principle of *countable choice* AC_0, namely choice for statements of the form $\forall\, n \in \mathbb{N} \,.\, \exists\, x \in X \,.\, \varphi$. We shall not need dependent choice. In fact, we mostly require only number-number choice, $\mathsf{AC}_{0,0}$, i.e. countable choice in the special case $X = \mathbb{N}$. However, in Lemmas 3.7 and 3.16, we make use of the stronger principle $\mathsf{AC}_{0,1}$, i.e. countable choice in the case $X = \mathbb{N}^{\mathbb{N}}$. We also assume extensionality for functions: if $f(x) = g(x)$ for all x then $f = g$, and if $x = y$ then $f(x) = f(y)$. (This is needed in Proposition 4.7 and in Corollary 4.8.) For the development that follows, it does not matter whether real numbers $\mathbb{R}$ are taken to be Cauchy sequences of rationals, with equality as an equivalence relation over them, or whether real numbers are taken to be equivalence classes of Cauchy sequences. The former is Bishop's approach to real numbers, the latter is the natural approach when reasoning in the internal logic of an elementary topos, where, because we assume $\mathsf{AC}_{0,0}$, the object $\mathbb{R}$ of equivalence classes of Cauchy sequences is isomorphic to the favoured object of Dedekind reals.

We assume familiarity with the constructive notions of metric space, Cauchy sequence (n.b. constructively a Cauchy sequence is required to have an associated modulus function) and convergence. Because we consider several notions of continuity, we spell out each one of them. A function $f : X \to Y$ between metric spaces is:

– *uniformly continuous* when for every $\varepsilon > 0$ there exists $\delta > 0$ such that, for all $x, x' \in X$, if $d(x, x') < \delta$ then $d(f(x), f(x')) < \varepsilon$.

– *pointwise continuous at* $x \in X$ when for every $\varepsilon > 0$ there exists $\delta > 0$ such that, for all $x' \in X$, $d(x, x') < \delta$ implies $d(f(x), f(x')) < \varepsilon$. A function which is pointwise continuous at every point is *pointwise continuous*.

– *sequentially continuous* when it preserves limits of convergent sequences: if $(a_i)_{i \in \mathbb{N}}$ converges to a in X then $(f(a_i))_{i \in \mathbb{N}}$ converges to $f(a)$ in Y.

Obviously, uniform continuity implies pointwise continuity, which in turn implies sequential continuity. Constructively, the converses need not hold, even for closed totally bounded spaces. Indeed, in Recursive Mathematics there exists a pointwise continuous function $[0, 1] \to \mathbb{R}$ which is unbounded and hence not uniformly continuous [20, §6.4.4]. Also, in the setting of Example 4.12 below, all functions $[0, 1] \to \mathbb{R}$ are sequentially continuous, but not all such functions are pointwise continuous.

For a metric space (X, d), we write $B(x, r)$ for the *open ball* centered at $x \in X$ with radius $r > 0$. We say that (X, d) is *separable* if it contains a countable dense subspace; and that it is *complete* if every Cauchy sequence converges. As is customary we abbreviate *complete separable metric space* as *CSM*.

For reference we list several standard examples of complete separable metric spaces. The set of real numbers $\mathbb{R}$ equipped with the usual metric $d(x, y) = |x - y|$ is a CSM, and so is the set of real sequences $\mathbb{R}^{\mathbb{N}}$ with the metric defined by

$$d(x, y) = \sum_{k=0}^{\infty} \min(1, |x_k - y_k|) \cdot 2^{-(k+1)} .$$

Baire space is defined as the subspace $\mathbb{Z}^{\mathbb{N}}$ of $\mathbb{R}^{\mathbb{N}}$ consisting of all integer sequences, and *Cantor space* $2^{\mathbb{N}}$ is the subspace $2^{\mathbb{N}} \subseteq \mathbb{Z}^{\mathbb{N}}$ of binary sequences:

$$2^{\mathbb{N}} = \{\alpha \in \mathbb{Z}^{\mathbb{N}} \mid \forall\, i \in \mathbb{N} \,.\, (\alpha_i = 0 \lor \alpha_i = 1)\} .$$

The *one-point compactification of* $\mathbb{N}$ is the subspace $\mathbb{N}^+$ of $2^{\mathbb{N}}$ defined by

$$\mathbb{N}^+ = \{\alpha \in 2^{\mathbb{N}} \mid \forall\, i \in \mathbb{N} \,.\, (\alpha_i = 0 \implies \forall\, j > i \,.\, \alpha_j = 0)\} .$$

$\mathbb{N}$ is a subspace of $\mathbb{N}^+$ where a number n is represented by the sequence κ_n whose first n terms are 1, and all others are 0. The space $\mathbb{N}^+$ also contains the *point at infinity* κ_∞ which is the constant 1 sequence.

An ε*-net* in a metric space X is a finite subset $N \subseteq X$ such that, for every $x \in X$, there exists $y \in N$ for which $d(x, y) < \varepsilon$. A CSM is said to be *complete totally bounded (CTB)* if it has an ε-net for every $\varepsilon > 0$. Cantor space, closed intervals $[a, b] = \{x \in \mathbb{R} \mid a \le x \le b\}$ and the space $\mathbb{N}^+$ are easily seen to be CTB. The notion of CTB space provides one possible constructive formulation of compactness, though, in general, it cannot be shown constructively that CTB spaces have the Heine-Borel property. Indeed, it is consistent for the Heine-Borel property to fail for Cantor space and $[0, 1]$.

We next define the concepts needed to formulate our main results. A point $x \in X$ is a *cluster point* if every open ball centered at x contains a point distinct from x. In the presence of $\mathsf{AC}_{0,0}$ this is equivalent to x being the limit of an *injective sequence* $(a_i)_{i \in \mathbb{N}}$, which is a sequence for which $d(a_n, a_m) > 0$ whenever $n \neq m$.

Definition 2.1 A metric space is *without isolated points* if every point is a cluster point.

We say that a sequence $(a_i)_{i \in \mathbb{N}}$ in a metric space (X, d) is *without accumulation point* when, for every $x \in X$, there exist $\varepsilon > 0$ and $m \in \mathbb{N}$ such that $d(x, a_i) > \varepsilon$ for all $i \ge m$.

Definition 2.2 A metric space (X, d) is *locally non-compact at* $x \in X$ if for every $\varepsilon > 0$ the open ball $B(x, \varepsilon)$ contains a sequence without accumulation point in X. It is *locally non-compact* if it is locally non-compact at every x.

Examples of spaces without isolated points are metric vector spaces, locally non-compact spaces, and dense subspaces of spaces without isolated points.

Any infinite-dimensional separable Hilbert space is a locally non-compact CSM, for example the space ℓ^2 of square-summable sequences; or the space $\mathcal{C}_{\mathsf{u}}[0, 1]$ of uniformly continuous maps $[0, 1] \to \mathbb{R}$, equipped with the supremum norm. The latter example generalizes as follows. If X is a CTB space and $f : X \to \mathbb{R}$ is uniformly continuous, then its supremum norm $\|f\|_\infty = \sup\{f(x) \mid x \in X\}$ is well defined, and so $\mathcal{C}_{\mathsf{u}}(X)$ is a normed vector space. Then we have the following proposition.

Proposition 2.3 *If (X, d) is a CTB space which contains a cluster point then $\mathcal{C}_{\mathsf{u}}(X)$ equipped with the supremum norm is locally non-compact.*

Proof. As $\mathcal{C}_{\mathsf{u}}(X)$ is a normed vector space, it suffices to find in $\mathcal{C}_{\mathsf{u}}(X)$ a single bounded injective sequence $(f_i)_{i\in\mathbb{N}}$ without accumulation point. Let $b \in X$ be a cluster point and $(a_i)_{i\in\mathbb{N}}$ an injective sequence which converges to b. Without loss of generality we may assume that $d(b, a_n) > 0$ for all $n \in \mathbb{N}$. For each $n \in \mathbb{N}$ let $f_n : X \to \mathbb{R}$ be defined by $f_n(x) = \max(0, 1 - 2d(x, a_n)/d(b, a_n))$. Clearly, $\|f_n\|_\infty = 1$ and $\|f_n - f_m\|_\infty > 0$ if $n \neq m$. It remains to be shown that $(f_i)_{i\in\mathbb{N}}$ is without accumulation point. Consider any uniformly continuous map $g : X \to \mathbb{R}$. If $|g(b)| > 1/3$ then $\|g - f_n\|_\infty \geq |g(b) - f_n(b)| = |g(b)| > 1/3$ for all $n \in \mathbb{N}$. If $|g(b)| < 2/3$ then there exists $\varepsilon > 0$ such that $|g(x)| < 2/3$ for all $x \in B(b, \varepsilon)$. Because $(a_i)_{i\in\mathbb{N}}$ converges to b there is $m \in \mathbb{N}$ such that $a_n \in B(b, \varepsilon)$ for all $n \geq m$. Therefore for $n \geq m$ we get $\|g - f_n\|_\infty \geq |g(a_n) - f_n(a_n)| = |g(a_n) - 1| > 1/3$. □

Two other important examples of locally non-compact CSMs are the space $\mathbb{R}^{\mathbb{N}}$ of infinite sequences of real numbers and Baire space $\mathbb{Z}^{\mathbb{N}}$.

In the presence of the formalized Church's Thesis, CT_0 [20, §4.3], the real line $\mathbb{R}$ and the closed interval $[0, 1]$ give surprising examples of locally non-compact spaces. This is because CT_0 implies the existence of *strong Specker sequences* [20, §6.4.7], which are nothing but bounded monotone sequences of reals without accumulation point. This shows that, constructively, it is consistent to have a CSM that is simultaneously CTB and locally non-compact.

We now state the two main embedding-extension results of this paper. The proofs are deferred to Section 3.

Theorem 2.4 *If X is an inhabited CSM without isolated points then there exists $e : 2^{\mathbb{N}} \to X$ such that:*

1. *The map e is a uniformly continuous embedding with closed image.*
2. *For every sequentially continuous $f : 2^{\mathbb{N}} \to \mathbb{Z}$, there exists a sequentially continuous $\overline{f} : X \to \mathbb{R}$ such that $f(x) = \overline{f}(e(x))$ for all $x \in X$.*

Theorem 2.5 *If X is an inhabited and locally non-compact CSM then there exists $e : \mathbb{Z}^{\mathbb{N}} \to X$ such that:*

1. *The map e is a uniformly continuous embedding with closed image.*
2. *For every sequentially continuous $f : \mathbb{Z}^{\mathbb{N}} \to \mathbb{Z}$, there exists a sequentially continuous $\overline{f} : X \to \mathbb{R}$ such that $f(x) = \overline{f}(e(x))$ for all $x \in X$.*

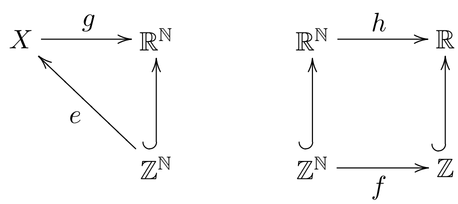
In classical mathematics, the second statement of each theorem follows from the first, as a consequence of the Tietze extension theorem [6, Theorem 7.5.1]. However, existing constructive versions of the Tietze Theorem, see e.g. [4, 5], are too restrictive to imply the results above.

Another difference from the classical setting is that, constructively, the different notions of continuity need not agree. Thus one can imagine analogues of the above results in which sequential continuity is replaced with other (stronger) forms of continuity. In fact, we have verified that our proofs of Theorems 2.4 and 2.5 adapt to show that, when the f above are pointwise continuous then so are the associated $\overline{f}$. Nevertheless, in the present paper, we restrict attention to the sequentially continuous versions stated above. Not only are the sequential versions the ones needed for our applications later on, but also their proofs turn out to be harder than the proofs when stronger notions of continuity are used.

3 The Proof of Theorems 2.4 and 2.5

The proofs of Theorems 2.4 and 2.5 are very similar. We therefore treat the (more interesting) case of Theorem 2.5 in detail, and afterwards outline how the proof may be adapted to deal with the (much easier) case of Theorem 2.4. We break Theorem 2.5 into two separate propositions.

Proposition 3.1 *For any inhabited locally non-compact CSM X, there exist a uniformly continuous embedding $e : \mathbb{Z}^{\mathbb{N}} \to X$ with closed image and a pointwise continuous map $g : X \to \mathbb{R}^{\mathbb{N}}$ such that the left-hand diagram below commutes.*



Proposition 3.2 *For every sequentially continuous $f : \mathbb{Z}^\mathbb{N} \to \mathbb{Z}$ there exists a sequentially continuous function $h : \mathbb{R}^\mathbb{N} \to \mathbb{R}$ such that the right-hand diagram above commutes.*

Theorem 2.5 follows immediately from Propositions 3.1 and 3.2, with the map $\overline{f} = h \circ g$ as the required extension of f along e.

3.1 Proof of Proposition 3.1

Throughout this section we assume that X is an inhabited locally non-compact CSM with a countable dense subset $S \subseteq X$. For the construction of g we will need the "cone" and "hill" functions, which we define now. For $x \in X$ and $0 < q < r$ let $\mathsf{cone}(x, r) : X \to \mathbb{R}$ and $\mathsf{hill}(x, r, q) : X \to \mathbb{R}$ be defined as

$$\mathsf{cone}(x, r)(y) = \max(0, 1 - r^{-1} \cdot d(x, y))) \,,$$
$$\mathsf{hill}(x, q, r)(y) = \max(0, 1 - (r - q)^{-1} \cdot \max(0, d(x, y) - q)) \,.$$

Lemma 3.3 *If $(a_i)_{i\in\mathbb{N}}$ and $(b_i)_{i\in\mathbb{N}}$ satisfy $\lim_{i\to\infty} d(a_i, b_i) = 0$ and if $(a_i)_{i\in\mathbb{N}}$ is a sequence without accumulation point then so is $(b_i)_{i\in\mathbb{N}}$.*

Proof. Consider an arbitrary $x \in X$. There exists $\varepsilon > 0$ and $m \in \mathbb{N}$ such that $d(x, a_i) > \varepsilon$ for all $i \geq m$. There exists $n \in \mathbb{N}$ such that $d(a_i, b_i) < \varepsilon/2$ for all $i \geq n$. Then for all $i \geq \max(m, n)$ we have $d(x, b_i) \geq d(x, a_i) - d(a_i, b_i) > \varepsilon/2$. □

Lemma 3.4 *Every open ball in X contains an injective sequence in S without accumulation point in X.*

Proof. Let $B(x, r)$ be an open ball in X. By assumption X is locally non-compact, so there exists a sequence $(a_i)_{i\in\mathbb{N}}$ in $B(x, r/2)$ without accumulation point. By $\mathsf{AC}_{0,0}$ there exists a sequence $(b_i)_{i\in\mathbb{N}}$ in S such that $d(a_i, b_i) < r/2^{i+1}$ for every $i \in \mathbb{N}$. By Lemma 3.3 the sequence $(b_i)_{i\in\mathbb{N}}$ is without accumulation point. It is contained in $B(x, r)$ because $d(x, b_i) \leq d(x, a_i) + d(a_i, b_i) < r/2 + r/2^{i+1} \leq r$.

By $\mathsf{AC}_{0,0}$ there is a choice function $f : \mathbb{N} \to \mathbb{N}$ which chooses for each $n \in \mathbb{N}$ some $f(n) > n$ such that there exists $\varepsilon > 0$ for which $d(b_n, b_m) > \varepsilon$ for all $m \geq f(n)$. Now the sequence $(c_i)_{i\in\mathbb{N}}$ defined by $c_n = b_{f^n(0)}$ is injective. Because it is a subsequence of $(b_i)_{i\in\mathbb{N}}$ it has no accumulation points and is contained in both S and $B(x, r)$, as required. □

We say that a sequence $(\varepsilon_i)_{i\in\mathbb{N}}$ of positive reals *convergently spaces* a sequence $(a_i)_{i\in\mathbb{N}}$ in X if $\lim_{i\to\infty} \varepsilon_i = 0$ and $i \neq j$ implies $d(a_i, a_j) > 2(\varepsilon_i + \varepsilon_j)$. Clearly, if $(\varepsilon_i)_{i\in\mathbb{N}}$ convergently spaces $(a_i)_{i\in\mathbb{N}}$ then $(a_i)_{i\in\mathbb{N}}$ is an injective sequence. For sequences without accumulation point, there is a converse.

Lemma 3.5 *If $(a_i)_{i\in\mathbb{N}}$ is an injective sequence without accumulation point, then for any $\varepsilon > 0$, there exists a sequence $(\varepsilon_i)_{i\in\mathbb{N}}$ of positive rationals $< \varepsilon$ that convergently spaces $(a_i)_{i\in\mathbb{N}}$.*

Proof. We show that, for all i, there exists a positive rational $\varepsilon_i < \min(2^{-i}, \varepsilon)$ such that, for all $j \neq i$, it holds that $\varepsilon_i < d(a_i, a_j)/4$. The lemma then follows immediately by $\mathsf{AC}_{0,0}$.

As $(a_i)_{i\in\mathbb{N}}$ is without accumulation point, for each i there exists m_i and $\zeta_i > 0$ such that, for all $j \geq m_i$, it holds that $d(a_i, a_j) > \zeta_i$. Because $(a_i)_{i\in\mathbb{N}}$ is injective, the value $\xi_i = \min\{d(a_i, a_j) \mid j < m_i \wedge j \neq i\}$ is positive. Thus there exists a positive rational $\varepsilon_i < \min(2^{-i}, \varepsilon, \zeta_i/4, \xi_i/4)$, and this has the required properties. □

Lemma 3.6 *If $(a_i)_{i\in\mathbb{N}}$ is without accumulation point and $(\varepsilon_i)_{i\in\mathbb{N}}$ convergently spaces $(a_i)_{i\in\mathbb{N}}$ then, for all $x \in X$, (i) there exists a unique k such that $d(x, a_k) < 2\varepsilon_k$, or (ii) for all i, it holds that $d(x, a_i) > \varepsilon_i$.*

Proof. As $(a_i)_{i\in\mathbb{N}}$ is without accumulation point, there exist m and $\zeta > 0$ such that, for all $j \geq m$ it holds that $d(x, a_j) > \zeta$. As $\lim_{i\to\infty} \varepsilon_i = 0$, there exists m' such that, for all $j \geq m'$, it holds that $\varepsilon_j < \zeta$. Thus, for all $j \geq \max(m, m')$, we have $d(x, a_j) > \varepsilon_j$. For each $i < \max(m, m')$, $d(x, a_i) < 2\varepsilon_i$ or $d(x, a_i) > \varepsilon_i$. As there are only finitely many $i < \max(m, m')\}$ (i) there exits $k < \max(m, m')$ such that $d(x, a_k) < 2\varepsilon_k$; or (ii) for all $i < \max(m, m')$, it holds that $d(x, a_i) > \varepsilon_i$. In the first case, the k such that $d(x, a_k) < 2\varepsilon_i$ is unique because $(\varepsilon_i)_{i\in\mathbb{N}}$ convergently spaces $(a_i)_{i\in\mathbb{N}}$. In the second case, $d(x, a_i) > \varepsilon_i$ for all i. □

If $(\varepsilon_i)_{i\in\mathbb{N}}$ convergently spaces $(a_i)_{i\in\mathbb{N}}$, then we say that $((a_i)_{i\in\mathbb{N}}, (\varepsilon_i)_{i\in\mathbb{N}})$ is *well inside* $B(v, \eta)$ if, for all i, it holds that $d(v, a_i) < \eta/3$ and $\varepsilon_i < \eta/3$.

Let $\mathbb{N}^*$ be the set of finite sequences of integers. If $a \in \mathbb{N}^*$ and $j \in \mathbb{N}$, we write aj for the sequence a followed by j. The empty sequence is denoted by $[\,]$ and the length of a is denoted by $|a|$. We write $\alpha\restriction_n$ for the prefix in $\mathbb{N}^n$ of an infinite sequence $\alpha \in \mathbb{N}^{\mathbb{N}}$.

Lemma 3.7 *There exist a family $(w(a))_{a\in\mathbb{N}^*}$ in S and a family $(\delta(a))_{a\in\mathbb{N}^*}$ of positive rational numbers such that, for every $a \in \mathbb{N}^*$:*

1. *the sequence $(w(ai))_{i\in\mathbb{N}}$ is without accumulation point;*
2. *the sequence $(\delta(ai))_{i\in\mathbb{N}}$ convergently spaces $(w(ai))_{i\in\mathbb{N}}$; and*
3. *$((w(ai))_{i\in\mathbb{N}}, (\delta(ai))_{i\in\mathbb{N}})$ is well inside $B(w(a), \delta(a))$.*

Proof. Given $v \in S$ and rational $\eta > 0$, we have by Lemmas 3.4 and 3.5 that there there exist an injective sequence $(v_i)_{i\in\mathbb{N}}$ in $S \cap B(v, \eta/3)$ without accumulation point, and a sequence $(\eta_i)_{i\in\mathbb{N}}$ of positive rationals $< \eta/3$ that convergently spaces $(v_i)_{i\in\mathbb{N}}$. As S is countable, by $\mathsf{AC}_{0,1}$, we obtain a function mapping each pair (v, η) to such a pair of sequences $((v_i)_{i\in\mathbb{N}}, (\eta_i)_{i\in\mathbb{N}})$.

To prove the lemma, start off by fixing $w([\,])$ to be any member of S and $\delta([\,]) = 1$. Then $(w(ai), \delta(ai))$ is defined by applying the above function to the pair $(w(a), \delta(a))$, and extracting the i-indexed components of the resulting sequences. It is immediate from the definition that this gives rise to families $(\delta(a))_{a\in\mathbb{N}^*}$ and $(w(a))_{a\in\mathbb{N}^*}$ satisfying the required properties. □

Lemma 3.8 *Let $(w(a))_{a\in\mathbb{N}^*}$ and $(\delta(a))_{a\in\mathbb{N}^*}$ be as in Lemma 3.7. If $a \in \mathbb{N}^*$ is a proper prefix of $b \in \mathbb{N}^*$ then $B(w(b), \delta(b)) \subseteq B(w(a), 2\delta(a)/3)$.*

Proof. This follows easily from property 3 of Lemma 3.7. □

Lemma 3.9 *Let $(w(a))_{a\in\mathbb{N}^*}$ and $(\delta(a))_{a\in\mathbb{N}^*}$ be as in Lemma 3.7. For all $a, b \in \mathbb{N}^n$ with $a \neq b$, it holds that $d(w(a), w(b)) > 2(\delta(a) + \delta(b))$.*

Proof. We can write $a \neq b$ as $a = cia'$ and $b = cjb'$, where c is the common prefix and $i \neq j$. The proof is by induction on $|a'| = |b'|$. When $|a'| = 0$ the lemma is immediate from property 2 of Lemma 3.7. For $|a'| > 0$, we have $a' = a''m$ and $b' = b''n$. The induction hypothesis gives $d(w(cia''), w(cjb'')) > 2(\delta(cia'') + \delta(cjb''))$. Also, by Lemma 3.8, we have $d(w(a), w(cia'')) < 2\delta(cia'')/3$ and $d(w(b), w(cjb'')) < 2\delta(cjb'')/3$. Thus $d(w(a), w(b)) > 4(\delta(cia'') + \delta(cjb''))/3$. However, by property 3 of Lemma 3.7, $\delta(a) < \delta(cia'')/3$ and $\delta(b) < \delta(cjb'')/3$. So $d(w(a), w(b)) > 4(\delta(a) + \delta(b)) > 2(\delta(a) + \delta(b))$. □

Lemma 3.10 *Let $(w(a))_{a\in\mathbb{N}^*}$ and $(\delta(a))_{a\in\mathbb{N}^*}$ be as in Lemma 3.7. For every $n \in \mathbb{N}$ and $x \in X$, there exists a unique $b \in \mathbb{N}^n$ such that $d(w(b), x) < 2\delta(b)$, or for all $a \in \mathbb{N}^n$ it holds that $d(w(a), x) > \delta(a)$.*

Proof. By induction on $n \in \mathbb{N}$. When $n = 0$ the lemma states that $d(w([\,]), x) < 2\delta([\,])$ or $d(w([\,]), x) > \delta([\,])$, which of course holds. Suppose $n > 0$. By Lemma 3.8, for all $a' \in \mathbb{N}^{n-1}$, if $d(x, w(a')) > \delta(a)$ then $d(x, w(a'i)) > \delta(a'i)$ for all i. By the induction hypothesis, there exists a unique $b' \in \mathbb{N}^{n-1}$ such that $d(x, w(b')) < 2\delta(b')$, or, for all $a' \in \mathbb{N}^{n-1}$, it holds that $d(x, w(a')) > \delta(a')$. In the second case, we are done by the previous observation. Thus suppose $b' \in \mathbb{N}^{n-1}$ is the unique such that $d(x, w(b')) < 2\delta(b')$. By uniqueness, for $a' \in \mathbb{N}^n$ with $a' \neq b'$, it holds that $d(x, w(a')) > \delta(a')$. Hence, by the observation above, $d(x, w(a'i)) > \delta(a'i)$, for all $a' \neq b' \in \mathbb{N}^{n-1}$ and i. So, if also $d(w(b'j), x) > \delta(b'j)$ for all j, then indeed $d(w(a), x) > \delta(a)$ for all $a \in \mathbb{N}^n$. By Lemma 3.6, the only other possibility is that there exists (a unique) k such that $d(w(b'k), x) < 2\delta(b'k)$. By Lemma 3.9, this is the unique $b \in \mathbb{N}^n$ with $d(w(b), x) < 2\delta(b)$. □

We now prove Proposition 3.1. Let $(w(a))_{a\in\mathbb{N}^*}$ and $(\delta(a))_{a\in\mathbb{N}^*}$ be as in (the proof of) Lemma 3.7, with $\delta([\,]) = 1$. Using any bijection between $\mathbb{Z}$ and $\mathbb{N}$, we rewrite these families as $(w(a))_{a\in\mathbb{Z}^*}$ and $(\delta(a))_{a\in\mathbb{Z}^*}$. Define the map $e : \mathbb{Z}^{\mathbb{N}} \to X$ by

$$e(\alpha) = \lim_{i\to\infty} w(\alpha\upharpoonright_i)\ .$$

This is well defined because, by statement 3 of Lemma 3.7, $\delta(a) \leq 3^{-|a|}$ for all $a \in \mathbb{Z}^*$, and so, by Lemma 3.8, the sequence $(w(\alpha\upharpoonright_i))_{i\in\mathbb{N}}$ is Cauchy. Also by Lemma 3.8, $d(e(\alpha), w(\alpha\upharpoonright_i)) \leq 2\delta(\alpha\upharpoonright_i)/3 < \delta(\alpha\upharpoonright_i) \leq 3^{-i}$. Thus e is uniformly continuous because, for any $\varepsilon > 0$, take $\delta = 2^{-k}$, where k is such that $3^{-k} < \varepsilon/2$. If $d(\alpha, \beta) < \delta$ then $\alpha\upharpoonright_k = \beta\upharpoonright_k$, so indeed $d(e(\alpha), e(\beta)) \leq d(e(\alpha), w(\alpha\upharpoonright_k)) + d(e(\beta), w(\beta\upharpoonright_k)) \leq 2\cdot 3^{-k} < \varepsilon$.

To define $g : X \to \mathbb{R}^{\mathbb{N}}$, we first define functions $g_i : X \to \mathbb{R}$, for each $i \in \mathbb{N}$, by:

$$g_i(x) = \begin{cases} a_i \cdot \mathsf{hill}(w(a), 2\delta(a)/3, \delta(a))(x) & \text{if there exists unique } a \in \mathbb{Z}^{i+1} \text{ with } d(w(a), x) < 2\delta(a), \\ 0 & \text{if } d(w(a), x) > \delta(a) \text{ for all } a \in \mathbb{Z}^{i+1}, \end{cases}$$

where we write any $a \in \mathbb{Z}^{i+1}$ as $a_0a_1 \ldots a_i$. The function g_i is well defined because when both clauses apply they agree that $g_i(x) = 0$, and, by Lemma 3.10, at least one of the cases always applies. Easily, when the first clause applies, then $g_i(y) = a_i \cdot \mathsf{hill}(w(a), 2\delta(a)/3, \delta(a))(y)$ for all $y \in B(x, 2\delta(a) - d(w(a), x))$. Similarly, when the second clause applies, $g_i(y) = 0$ for all $y \in B(x, d(w(a), x) - \delta(a))$. Thus the function g_i is pointwise continuous. Now define $g : X \to \mathbb{R}^{\mathbb{N}}$ by $g(x) = (g_i(x))_{i\in\mathbb{N}}$. This is also pointwise continuous because the metric on $\mathbb{R}^{\mathbb{N}}$ defines $\mathbb{R}^{\mathbb{N}}$ as a countable product with respect to pointwise continuous maps.

By Lemma 3.8, for any $\alpha = \alpha_0\alpha_1 \ldots \in \mathbb{Z}^{\mathbb{N}}$, we have $d(w(\alpha\upharpoonright_{i+1}), e(\alpha)) \leq 2\delta(\alpha\upharpoonright_{i+1})/3 < 2\delta(\alpha\upharpoonright_{i+1})$, so $g_i(e(\alpha)) = \alpha_i \cdot \mathsf{hill}(w(\alpha\upharpoonright_{i+1}), 2\delta(\alpha\upharpoonright_{i+1})/3, \delta(\alpha\upharpoonright_{i+1}))(e(\alpha)) = \alpha_i$. Therefore $g(e(\alpha)) = \alpha$.

It remains to show that e is injective and that its image is closed. It is injective because $g \circ e$ is injective. To see that the image is closed, consider a sequence $(\alpha_i)_{i\in\mathbb{N}}$ in $\mathbb{Z}^{\mathbb{N}}$ such that $(e(\alpha_i))_{i\in\mathbb{N}}$ converges to $x \in X$. Because g is pointwise continuous the sequence $(g(e(\alpha_i)))_{i\in\mathbb{N}} = (\alpha_i)_{i\in\mathbb{N}}$ converges to $g(x)$, where $g(x) \in \mathbb{Z}^{\mathbb{N}}$ because $\mathbb{Z}^{\mathbb{N}}$ is a closed subspace of $\mathbb{R}^{\mathbb{N}}$. Therefore, x and $e(g(x))$ are both limits of $(e(\alpha_i))_{i\in\mathbb{N}}$, hence equal, and so x is in the image of e. This concludes the proof of Proposition 3.1.

3.2 Proof of Proposition 3.2

For the proof of Proposition 3.2, assume given a sequentially continuous $f : \mathbb{Z}^{\mathbb{N}} \to \mathbb{Z}$. We construct a function $h : \mathbb{R}^{\mathbb{N}} \to \mathbb{R}$ extending f.

For $\gamma \in \mathbb{R}^{\mathbb{N}}$ and $\beta \in \mathbb{Z}^{\mathbb{N}}$, define a sequence $(h^\beta_i(\gamma))_{i\in\mathbb{N}}$ of real numbers by:

$$h^\beta_0(\gamma) = f(0^\omega)\ ,$$

$$h^\beta_{i+1}(\gamma) = h^\beta_i(\gamma) + (f(\beta\upharpoonright_{i+1}0^\omega) - h^\beta_i(\gamma)) \cdot \prod_{j=0}^{i} \left(\mathsf{cone}(\beta_j, 1/4)(\gamma_j)\right)^{2^{i-j}}\ .$$

We say that β is *adequate for* γ if, for all $i \in \mathbb{N}$,

$$\beta_i - 2/3 < \gamma_i < \beta_i + 2/3\ .$$

By $\mathsf{AC}_{0,0}$, for every $\gamma \in \mathbb{R}^{\mathbb{N}}$, there exists $\beta \in \mathbb{Z}^{\mathbb{N}}$ adequate for γ.

Lemma 3.11 *If β and β' are both adequate for γ then $h^\beta_i(\gamma) = h^{\beta'}_i(\gamma)$.*

P r o o f. The proof proceeds by induction on i. Clearly $h^\beta_i(\gamma) = h^{\beta'}_i(\gamma)$ in the case that $\beta_j = \beta'_j$, for all $j < i$. Otherwise, without loss of generality, there exists $j < i$ such that $\beta_j < \beta'_j$. Then, as both β and β' are adequate for γ, it holds that $\gamma_j - 2/3 < \beta_j < \beta'_j < \gamma_j + 2/3$. Thus $\beta'_j = \beta_j + 1$ and $\beta_j + 1/3 < \gamma_j < \beta'_j - 1/3$, so $\mathsf{cone}(\beta_j, 1/4)(\gamma_j) = 0 = \mathsf{cone}(\beta'_j, 1/4)(\gamma_j)$. By induction hypothesis, $h^\beta_i(\gamma) = h^\beta_{i-1}(\gamma) = h^{\beta'}_{i-1}(\gamma) = h^{\beta'}_i(\gamma)$. □

The above lemma justifies the definition

$$h_i(\gamma) = h_i^\beta(\gamma), \text{ for any } \beta \text{ adequate for } \gamma\ .$$

The following technical lemma is in preparation for Lemma 3.13 below.

Lemma 3.12 *Let $(\xi_i)_{i\in\mathbb{N}}$ be a sequence in $[0,1]$ satisfying $\xi_{i+1} \leq \xi_i^2$, for all $i \in \mathbb{N}$, and let $P_i = \prod_{j=0}^{i-1}(1-\xi_j)$. Then $m \geq n$ implies $|P_n - P_m| \leq (2/3)^n$, and so the infinite product $\prod_{j=0}^{\infty}(1-\xi_j) = \lim_{i\to\infty} P_i$ converges.*

P r o o f. We show that $m \geq n$ implies $P_n - P_m \leq (2/3)^n$ (and also obviously $0 \leq P_n - P_m$). There are two cases. First, if $\xi_i > 1/3$ for all $i < n$, then

$$P_n - P_m \leq P_n \leq (2/3)^n\ .$$

In the second case there exists $k < n$ such that $\xi_k < 1/2$ and $\xi_i > 1/3$ for all $i < k$. Then, for all $i \geq k$, $P_i \leq P_k \leq (2/3)^k$ and $\xi_i < (1/2)^{2^{i-k}}$, so

$$P_i - P_{i+1} = P_i \cdot \xi_i < (2/3)^k \cdot (1/2)^{2^{i-k}} \leq (2/3)^k \cdot (1/2)^{1+i-k}\ .$$

From this we derive

$$P_n - P_m < (2/3)^k \cdot \sum_{i=n}^{m-1} (1/2)^{1+i-k} < (2/3)^k \cdot (1/2)^{n-k} \leq (2/3)^n\ .$$

□

Lemma 3.13 *For every $\gamma \in \mathbb{R}^{\mathbb{N}}$, the sequence $(h_i(\gamma))_{i\in\mathbb{N}}$ converges.*

P r o o f. Let β be adequate for γ. We must show that $(h_i^\beta(\gamma))_{i\in\mathbb{N}}$ converges. As f is sequentially continuous, there exists n such that $f(\beta{\upharpoonright}_m 0^\omega) = f(\beta)$ for all $m \geq n$. Then, for $m \geq n$, the equality

$$h_m^\beta(\gamma) = f(\beta) + (h_n^\beta(\gamma) - f(\beta)) \cdot \prod_{i=n}^{m-1}\left(1 - \prod_{j=0}^{i}\left(\mathsf{cone}(\beta_j, 1/4)(\gamma_j)\right)^{2^{i-j}}\right) \tag{1}$$

is easily shown by induction on m. Define

$$\xi_k(\beta,\gamma) = \prod_{j=0}^{n+k}\left(\mathsf{cone}(\beta_j, 1/4)(\gamma_j)\right)^{2^{n+k-j}}\ . \tag{2}$$

By Lemma 3.12, $\xi(\beta,\gamma) = \prod_{k=0}^{\infty}(1-\xi_k(\beta,\gamma))$ exists, and so the sequence $(h_i^\beta(\gamma))_{i\in\mathbb{N}}$ converges to $f(\beta) + (h_n^\beta(\gamma) - f(\beta)) \cdot \xi(\beta,\gamma)$ □

We define h to be the function

$$h(\gamma) = \lim_{i\to\infty} h_i(\gamma)\ . \tag{3}$$

By the proof of the above lemma, if β is adequate for γ and n is such that $f(\beta{\upharpoonright}_m 0^\omega) = f(\beta)$ for all $m \geq n$, then

$$h(\gamma) = f(\beta) + (h_n^\beta(\gamma) - f(\beta)) \cdot \xi(\beta,\gamma)\ . \tag{4}$$

Lemma 3.14 *For all $\alpha \in \mathbb{Z}^{\mathbb{N}}$, it holds that $h(\alpha) = f(\alpha)$.*

P r o o f. Trivially, $\beta = \alpha$ is the only sequence adequate for α. We must show that $\lim_{m\to\infty} h_m^\alpha(\alpha) = f(\alpha)$. Let n be such that, for all $m \geq n$, it holds that $f(\alpha{\upharpoonright}_m 0^\omega) = f(\alpha)$. Then, by (1), we have $h_m^\alpha(\gamma) = f(\alpha)$ for all $m > n$, because $\mathsf{cone}(\alpha_j, 1/4)(\alpha_j) = 1$ for all j. □

It remains to show that h is sequentially continuous. This result is not needed for Sections 4 and 5. The readers who are mostly interested in the last two sections may wish to skip the following proof.

Observe that the functions $\xi_k : \mathbb{Z}^\mathbb{N} \times \mathbb{R}^\mathbb{N} \to \mathbb{R}$ defined in (2) are uniformly continuous because they are finite products of cone functions. By Lemma 3.12, the product $\xi(\beta, \gamma) = \prod_{k=0}^{\infty}(1 - \xi_k(\beta, \gamma))$ converges uniformly. Therefore, $\xi : \mathbb{Z}^\mathbb{N} \times \mathbb{R}^\mathbb{N} \to \mathbb{R}$ is a pointwise continuous function. Similarly, for any $i \in \mathbb{N}$ and $\beta \in \mathbb{Z}^\mathbb{N}$, the function $h_i^\beta : \mathbb{R}^\mathbb{N} \to \mathbb{R}$ is uniformly continuous, because it is a polynomial of cone functions. However, h need not be pointwise continuous, because its defining limit (3) is not necessarily uniform.

Lemma 3.15 *Suppose $(\gamma^i)_{i \in \mathbb{N}}$ is a sequence in $\mathbb{R}^\mathbb{N}$ converging to γ. There exists a sequence $(\beta^i)_{i \in \mathbb{N}}$ in $\mathbb{Z}^\mathbb{N}$ converging to β such that β is adequate for γ and β^i is adequate for γ^i for every $i \in \mathbb{N}$.*

P r o o f. By $\mathsf{AC}_{0,0}$ there exists $\beta \in \mathbb{Z}^\mathbb{N}$ such that, for all $i \in \mathbb{N}$,

$$\beta_i - 5/9 < \gamma_i < \beta_i + 5/9 \; .$$

By $\mathsf{AC}_{0,1}$ there exists a sequence $(\delta^i)_{i \in \mathbb{N}}$ in $\mathbb{Z}^\mathbb{N}$ such that δ^i is adequate for γ^i, for all $i \in \mathbb{N}$. Using $\mathsf{AC}_{0,0}$ and the fact that $(\gamma_j^i)_{i \in \mathbb{N}}$ converges to γ_j, we obtain a function $m : \mathbb{N} \to \mathbb{N}$ such that for all $j \in \mathbb{N}$ and $i \geq m(j)$ it holds that $|\gamma_j^i - \gamma_j| < 1/9$. Now define the sequence β^i by

$$\beta_j^i = \begin{cases} \beta_j & \text{if } i \geq m(j), \\ \delta_j^i & \text{otherwise.} \end{cases}$$

We claim that each β^i is adequate for γ^i. Indeed, if $i < m(j)$ then $\beta_j^i = \delta_j^i$, and if $i \geq m(j)$ then

$$|\beta_j^i - \gamma_j^i| = |\beta_j - \gamma_j^i| \leq |\beta_j - \gamma_j| + |\gamma_j^i - \gamma_j| < 5/9 + 1/9 = 2/3 \; .$$

Finally, $\lim_{i \to \infty} \beta^i = \beta$ because β agrees with β^i in the first j terms when $i \geq \max(m(0), \dots, m(j-1))$. □

Lemma 3.16 *Suppose $(\beta^i)_{i \in \mathbb{N}}$ converges to β in $\mathbb{Z}^\mathbb{N}$. There exists $k \in \mathbb{N}$ such that, for all $m, n \geq k$,*

$$f(\beta^m \restriction_n 0^\omega) = f(\beta^m) = f(\beta \restriction_n 0^\omega) = f(\beta) \; .$$

P r o o f. For $j \in \mathbb{N}$ define $L_j \subseteq \mathbb{N} \times \mathbb{N}$ to be $L_j = \{(m, j) \mid m \geq j\} \cup \{(j, n) \mid n \geq j\}$. By sequential continuity of f there exists $\ell \in \mathbb{N}$ such that, for all $j \geq \ell$, $f(\beta) = f(\beta^j) = f(\beta \restriction_j 0^\omega)$. We claim that, for all $j \geq \ell$, either

$$\forall\, (m, n) \in L_j \,.\, f(\beta^m \restriction_n 0^\omega) = f(\beta) \tag{5}$$

or

$$\exists\, (m, n) \in L_j \,.\, f(\beta^m \restriction_n 0^\omega) \neq f(\beta) \; . \tag{6}$$

To see this, use sequential continuity of f to obtain $j' \geq j$ such that, for all $m, n \geq j'$, $f(\beta^m \restriction_j 0^\omega) = f(\beta \restriction_j 0^\omega) = f(\beta)$ and $f(\beta^j \restriction_n 0^\omega) = f(\beta^j) = f(\beta)$. By inspecting the finitely many values $\{(m, n) \in L_j \mid m, n < j'\}$ it can now be determined whether (5) or (6) holds.

Next we define sequences $(m(j))_{j \geq \ell}$ and $(n(j))_{j \geq \ell}$ as follows:

$$(m(j), n(j)) = \begin{cases} (j, j) & \text{if (5) holds for } j, \\ (m, n) & \text{the lexicographically smallest } (m, n) \in L_j \text{ for which (6) holds.} \end{cases}$$

The sequence $(\beta^{m(j)} \restriction_{n(j)} 0^\omega)_{j \geq \ell}$ converges to β in $\mathbb{Z}^\mathbb{N}$ because $m(j) \geq j$ and $n(j) \geq j$. By the sequential continuity of f, there exists $k \geq \ell$ such that, for all $j \geq k$, $f(\beta^{m(j)} \restriction_{n(j)} 0^\omega) = f(\beta)$. By the definition of $(m(j), n(j))$, we must have (5) for all $j \geq k$. Therefore, if $m, n \geq k$ then $f(\beta^m \restriction_n 0^\omega) = f(\beta^m) = f(\beta \restriction_n 0^\omega) = f(\beta)$ because $(m, n) \in L_{\min(m,n)}$ and $\min(m, n) \geq k \geq \ell$. □

At last we show that h is sequentially continuous. Let $(\gamma^i)_{i\in\mathbb{N}}$ be a sequence in $\mathbb{R}^{\mathbb{N}}$ converging to γ. By Lemma 3.15 there exists a sequence $(\beta^i)_{i\in\mathbb{N}}$ in $\mathbb{Z}^{\mathbb{N}}$ converging to β, such that β is adequate for γ and each β^i is adequate for γ^i. By Lemma 3.16 there exists $k \in \mathbb{N}$ such that $f(\beta^m \restriction_n 0^\omega) = f(\beta^m) = f(\beta\restriction_n 0^\omega) = f(\beta)$ for all $m, n \geq k$. By (4), for all $m \geq k$,

$$\begin{aligned} h(\gamma^m) &= f(\beta^m) + (h_k^{\beta^m}(\gamma^m) - f(\beta^m)) \cdot \xi(\beta^m, \gamma^m) \\ &= f(\beta) + (h_k^{\beta^m}(\gamma^m) - f(\beta)) \cdot \xi(\beta^m, \gamma^m) \ . \end{aligned}$$

There exists $k' \in \mathbb{N}$ such that, for all $m \geq k'$, β and β^m agree in the first k terms. Thus, for $m \geq k'$, it holds that $h_k^{\beta^m}(\gamma^m) = h_k^{\beta}(\gamma^m)$, hence for all $m \geq \max(k', k)$ it is the case that

$$h(\gamma^m) = f(\beta) + (h_k^{\beta}(\gamma^m) - f(\beta)) \cdot \xi(\beta^m, \gamma^m) \ .$$

We observed above that the functions ξ and h_k^β are pointwise continuous, so

$$\begin{aligned} \lim_{m\to\infty} h(\gamma^m) &= \lim_{m\to\infty} (f(\beta) + (h_k^{\beta}(\gamma^m) - f(\beta)) \cdot \xi(\beta^m, \gamma^m)) \\ &= f(\beta) + (h_k^{\beta}(\gamma) - f(\beta)) \cdot \xi(\beta, \gamma) \\ &= h(\gamma) \ , \end{aligned}$$

where the last equality follows from (4), using $f(\beta\restriction_n 0^\omega) = f(\beta)$ for all $n \geq k$.

This completes the proof of Proposition 3.2. Observe that, in addition to showing the existence of h given f, the proof constructs a function mapping any sequentially continuous function $f : \mathbb{Z}^{\mathbb{N}} \to \mathbb{Z}$ to a corresponding sequentially continuous extension $h_f : \mathbb{R}^{\mathbb{N}} \to \mathbb{R}$.

Remark 3.17 Under the stronger assumption that f is uniformly continuous on CTB subspaces of $\mathbb{Z}^{\mathbb{N}}$ (this is the main notion of continuity used by Bishop [4]) there is an easier construction of an extension function h. For, any $x \in \mathbb{R}$ define the probability distribution $p_x \colon \mathbb{Z} \to [0, 1]$ by $p_x(a) = \max(0, 1 - \max(1, |x - a|))$. For any $\gamma \in \mathbb{R}^{\mathbb{N}}$ define μ_γ to be the product measure on $\mathbb{Z}^{\mathbb{N}}$ whose i-th component is the measure on $\mathbb{Z}$ determined by p_{γ_i}. Then define

$$h(\gamma) = \int f \, \mathrm{d}\mu_\gamma \ .$$

Constructively, the assumption that f is uniformly continuous on CTB subspaces is needed to ensure that the integral is well defined. A generalization of this approach to extending functionals has been worked out in a classical setting by Normann, who has embedded the entire continuous type hierarchy over $\mathbb{N}$ in the continuous type hierarchy over $\mathbb{R}$, see [17].

Remark 3.18 It should be possible to avoid the technical proof that h is sequentially continuous, by proving a meta-theorem guaranteeing that, because h is defined constructively from functions that are themselves sequentially continuous, it holds automatically that h is sequentially continuous too. One possible approach to formalizing such a meta-theorem would be to develop a constructive analogue of Johnstone's "topological topos" $\mathcal{T}$ of sheaves for the canonical Grothendieck topology on the monoid of continuous endomorphisms on $\mathbb{N}^+$ [12]. Then the relativization of the construction of h to $\mathcal{T}$ would result in a sequentially continuous function being produced. It would be interesting to see this worked out in detail.

Remark 3.19 The proof of Lemma 3.16 can be generalized to show that every sequentially continuous $f : \mathbb{N}^+ \times \mathbb{N}^+ \to \mathbb{N}$ is uniformly continuous. For further connections between sequential continuity and the space $\mathbb{N}^+$, see Section 4 below.

In Section 5, we shall use a couple of straightforward consequences of the proof of Proposition 3.2, rather than the result itself. To formulate these, we say that $G \subseteq \mathbb{R}^{\mathbb{N}}$ and $B \subseteq \mathbb{Z}^{\mathbb{N}}$ form an *adequate subdomain pair* if:

1. for all $\gamma \in G$ there exists $\beta \in B$ with β adequate for γ, and

2. for all $\beta \in B$ and $n \in \mathbb{N}$ it holds that $\beta_0 \dots \beta_{n-1} 0^\omega \in B$.

Given such G and B and sequentially continuous $f : B \to \mathbb{Z}$, the proof of Proposition 3.2 clearly constructs a function $h_{(G,B,f)} : G \to \mathbb{R}$.

Lemma 3.20 *If G and B are an adequate subdomain pair and $f : B \to \mathbb{Z}$ is sequentially continuous then $h_{(G,B,f)} : G \to \mathbb{R}$ satisfies $h_{(G,B,f)} = f(\alpha)$ for all $\alpha \in G \cap B$.*

P r o o f. Identical to the proof of Lemma 3.14. □

Lemma 3.21 *If G, B and G', B' are adequate subdomain pairs, and $f : B \to \mathbb{Z}$ and $f' : B' \to \mathbb{Z}$ are sequentially continuous functions satisfying $f(\alpha) = f'(\alpha)$ for all $\alpha \in B \cap B'$, then $h_{(G,B,f)}(\gamma) = h_{(G',B',f')}(\gamma')$ for all $\gamma \in G \cap G'$.*

P r o o f. Suppose $\gamma \in G \cap G'$. Let $\beta \in B$ and $\beta' \in B'$ be adequate for γ. Write $h_i^{f,\beta}(\gamma)$ for the convergent sequence determining $h_{(G,B,f)}(\gamma)$ and $h_i^{f',\beta'}(\gamma)$ for that determining $h_{(G',B',f')}(\gamma')$. Note, of course, that the function f is used in the recursive definition of $h_i^{f,\beta'}(\gamma)$, whereas f' is used in the definition of $h_i^{f',\beta'}(\gamma)$. We show by induction on i that $h_i^{f,\beta}(\gamma) = h_i^{f',\beta'}(\gamma)$.

Suppose $\beta_j = \beta'_j$, for all $j < i$. Then $\beta\restriction_i 0^\omega = \beta'\restriction_i 0^\omega \in B \cap B'$, so $f(\beta\restriction_i 0^\omega) = f'(\beta'\restriction_i 0^\omega)$. Thus, by applying the induction hypothesis to the definitions of $h_i^{f,\beta}(\gamma)$ and $h_i^{f',\beta'}(\gamma)$, we obtain $h_i^{f,\beta}(\gamma) = h_i^{f',\beta'}(\gamma)$.

Otherwise, without loss of generality, there exists $j < i$ such that $\beta_j < \beta'_j$. As in the proof of Lemma 3.11, $\mathsf{cone}(\beta_j, 1/4)(\gamma_j) = 0 = \mathsf{cone}(\beta'_j, 1/4)(\gamma_j)$. So, by induction hypothesis, $h_i^{f,\beta}(\gamma) = h_{i-1}^{f,\beta}(\gamma) = h_{i-1}^{f',\beta'}(\gamma) = h_i^{f',\beta'}(\gamma)$. □

3.3 Proof of Theorem 2.4

We conclude this section with a summary of the proof of Theorem 2.4. The main difference is to replace Proposition 3.1 with the analogous result below.

Proposition 3.22 *Let X be an inhabited CSM without isolated points. There exists a uniformly continuous embedding $e : 2^\mathbb{N} \to X$ with a closed image and a pointwise continuous map $g : X \to \mathbb{R}^\mathbb{N}$ such that $g(e(\alpha)) = \alpha$ for all $\alpha \in 2^\mathbb{N}$.*

The proof follows along the same general lines as that of Proposition 3.1, but is significantly simpler. In the proof of Proposition 3.1, a family $(B(w(a), \delta(a)))_{a \in \mathbb{N}^*}$ of open balls in X is defined with $B(w(b), \delta(b)) \subseteq B(w(a), 2\delta(a)/3)$ whenever a is a proper prefix of b, and such that each sequence $(w(ai))_{i \in \mathbb{N}}$ is without accumulation point. To prove Proposition 3.22, one more easily constructs a family $(B(w(a), \delta(a)))_{a \in 2^*}$ of open balls, again with $B(w(b), \delta(b)) \subseteq B(w(a), 2\delta(a)/3)$ whenever a is a proper prefix of b, but such that each sequence $w(a0) \neq w(a1)$. The required function g can then be defined in much the same way as before, but nowhere in the proof is there any need for analogues of Lemmas 3.3–3.6 and 3.10, which are all specific to sequences without accumulation point.

Finally, Theorem 2.4 is easily derived from a combination of Propositions 3.22 and 3.2. Indeed, given a sequentially continuous $f : 2^\mathbb{N} \to \mathbb{Z}$, this easily extends to a sequentially continuous $f' : \mathbb{Z}^\mathbb{N} \to \mathbb{Z}$, and hence Proposition 3.2 applies to yield a sequentially continuous $h \colon \mathbb{R}^\mathbb{N} \to \mathbb{R}$. Then the composite $\overline{f} = h \circ g$, where g is given by Proposition 3.22, gives the extension required by Theorem 2.4.

4 Applications to Continuity Principles

Continuity principles are statements asserting that all functions between certain spaces are continuous. Nontrivial continuity principles are inconsistent with classical mathematics, but play an important rôle in Brouwer's intuitionistic mathematics. They are also a feature of the internal logic of many toposes. In this section we apply Theorems 2.4 and 2.5 to derive new relationships between different continuity principles.

For metric spaces X and Y we consider the two continuity principles:

$\mathsf{CP}_{\mathsf{pt}}(X, Y)$: *All functions* $f : X \to Y$ *are pointwise continuous,*

$\mathsf{CP}_{\mathsf{seq}}(X, Y)$: *All functions* $f : X \to Y$ *are sequentially continuous.*

The sequence of propositions below, which is mostly folklore, summarizes basic relationships between the main continuity principles. In them, we write $\mathsf{AC}_{1,0}$ for the the principle of choice for statements of the form $\forall\, f \in \mathbb{N}^{\mathbb{N}} \,.\, \exists\, n \in \mathbb{N} \,.\, \varphi$.

Proposition 4.1 *Consider the following statements.*

1. $\mathsf{CP}_{\mathsf{pt}}(\mathbb{Z}^{\mathbb{N}}, \mathbb{N})$.
2. $\mathsf{CP}_{\mathsf{pt}}(X, \mathbb{N})$, *for all CSMs* X.
3. $\mathsf{CP}_{\mathsf{pt}}(\mathbb{Z}^{\mathbb{N}}, \mathbb{R})$.
4. $\mathsf{CP}_{\mathsf{pt}}(X, Y)$, *for all CSMs* X *and metric spaces* Y.

Then $1 \iff 2 \Longleftarrow 3 \iff 4$. *Moreover, if* $\mathsf{AC}_{1,0}$ *holds then* $1 \implies 3$.

P r o o f. The implication $2 \implies 1$ is immediate, and 4 implies the other statements. In [21, §7.2.7], it is shown that every CSM is a quotient (with respect to pointwise continuous maps) of $\mathbb{Z}^{\mathbb{N}}$. This gives the implication $1 \implies 2$. Similarly, if 3 holds then so does $\mathsf{CP}_{\mathsf{pt}}(X, \mathbb{R})$ for all CSMs X. To see that this implies 4, consider any $f : X \to Y$ and element $x \in X$. By $\mathsf{CP}_{\mathsf{pt}}(X, \mathbb{R})$, the function $x' \mapsto d(f(x), f(x')) : X \to \mathbb{R}$ is pointwise continuous. Hence, for any $\varepsilon > 0$ there exists δ such that, for all $x' \in B(x, \delta)$, we have $d(f(x), f(x')) < \varepsilon$. Thus f is continuous at every x, hence pointwise continuous. Finally if both 1 and $\mathsf{AC}_{1,0}$ hold then so does the principle WC-N, see [20, §4.6.3]. It is shown in [21, §7.2.7] that WC-N and 4 together imply $\mathsf{CP}_{\mathsf{pt}}(X, Y)$, for all CSMs X and separable metric spaces Y. Thus, in particular, 3 holds. □

Recall, from Section 2, the notion of CTB space. We say that a metric space X is *locally CTB* if every point in X has a CTB neighbourhood.

Proposition 4.2 *Consider the following statements.*

1. $\mathsf{CP}_{\mathsf{pt}}(2^{\mathbb{N}}, \mathbb{N})$.
2. $\mathsf{CP}_{\mathsf{pt}}(X, \mathbb{N})$, *for all locally CTB CSMs* X.
3. $\mathsf{CP}_{\mathsf{pt}}(2^{\mathbb{N}}, \mathbb{R})$.
4. $\mathsf{CP}_{\mathsf{pt}}(X, Y)$, *for all locally CTB CSMs* X *and metric spaces* Y.

Then $1 \iff 2 \Longleftarrow 3 \iff 4$. *Moreover, if* $\mathsf{AC}_{1,0}$ *holds then* $1 \implies 3$.

P r o o f. Similar to the proof of Proposition 4.1, making use of the fact that $2^{\mathbb{N}}$ is itself CTB (and hence locally CTB), and of the fact that every CTB space is a quotient of $2^{\mathbb{N}}$, which is established in [21, §7.4.4]. □

Proposition 4.4 below is an analogue of the preceding propositions for sequential continuity principles. First we show that, for maps out of the space $\mathbb{N}^+$ defined in Section 2, the sequential and pointwise continuity principles coincide.

Proposition 4.3 *For any metric space* Y, *the following are equivalent.*

1. $\mathsf{CP}_{\mathsf{pt}}(\mathbb{N}^+, Y)$.
2. $\mathsf{CP}_{\mathsf{seq}}(\mathbb{N}^+, Y)$.
3. *For all* $f : \mathbb{N}^+ \to Y$ *and* $\varepsilon > 0$ *there exists* $m \in \mathbb{N}$ *such that, for all* $n \geq m$, *we have* $d(f(\kappa_n), f(\kappa_\infty)) < \varepsilon$.

P r o o f. Trivially, $1 \implies 2 \implies 3$. Suppose then that 3 holds. We must establish $\mathsf{CP}_{\mathsf{pt}}(\mathbb{N}^+, Y)$. Consider any $f\colon \mathbb{N}^+ \to Y$, $\alpha \in \mathbb{N}^+$ and $\varepsilon > 0$. By 3, there exists $m \in \mathbb{N}$ such that, for all $n \geq m$, it holds that $d(f(\kappa_n), f(\kappa_\infty)) < \varepsilon/4$. We prove that, for $\delta = 2^{-(m+1)}$ and $\alpha' \in B(\alpha, \delta)$ it holds that $d(f(\alpha), f(\alpha')) < \varepsilon$.

If $i < m$ is the least such that $\alpha_i = 0$ then $\alpha = \kappa_i$, and $B(\alpha, 2^{-m+1}) \subseteq B(\alpha, 2^{-i+1}) = \{\alpha\}$. Thus, for all $\alpha' \in B(\alpha, 2^{-i+1})$ we have $d(f(\alpha), f(\alpha')) = 0 < \varepsilon$.

Otherwise, $\alpha_i = 1$ for all $i < m$. Define $g\colon \mathbb{N}^+ \to \mathbb{N}^+$ by $g(\beta)_i = \alpha_i \cdot \beta_i$. Then, for any $m' \geq m$, we have $g(\kappa_{m'}) = \kappa_n$ for a unique n with $m \leq n \leq m'$. Also, trivially, $g(\kappa_\infty) = \alpha$. So, by applying 3 to $g \circ f$, there exists $m' \in \mathbb{N}$ such that, for all $n' \geq m'$, we have $d(f(g(\kappa_{n'})), f(\alpha)) < \varepsilon/4$. Then, $g(\kappa_{\max(m,m')}) = \kappa_n$ for some n with $m \leq n \leq \max(m, m')$. So, $d(f(\kappa_n), f(\alpha)) < \varepsilon/4$, but also $d(f(\kappa_n), f(\kappa_\infty)) < \varepsilon/4$, because $n \geq m$. Thus $d(f(\alpha), f(\kappa_\infty)) < \varepsilon/2$.

Now consider any $\alpha' \in B(\alpha, 2^{-(m+1)})$. Then $\alpha'_i = 1$ for all $i < m$. By the same argument as above, $d(f(\alpha'), f(\kappa_\infty)) < \varepsilon/2$. Thus $d(f(\alpha), f(\alpha')) < \varepsilon$, as required. □

Proposition 4.4 *Consider the following statements.*

1. $\mathsf{CP}_{\mathsf{seq}}(\mathbb{N}^+, \mathbb{N})$.
2. $\mathsf{CP}_{\mathsf{seq}}(X, \mathbb{N})$, *for all CSMs* X.
3. $\mathsf{CP}_{\mathsf{seq}}(\mathbb{N}^+, \mathbb{R})$.
4. $\mathsf{CP}_{\mathsf{seq}}(X, Y)$, *for all CSMs* X *and metric spaces* Y.

Then $1 \iff 2 \impliedby 3 \iff 4$. *Moreover, if* $\mathsf{AC}_{1,0}$ *holds then* $1 \implies 3$.

P r o o f. Trivially $2 \implies 1$, and 4 implies the other statements. To show that 1 implies 2, suppose that 1 holds, and consider any $f\colon X \to \mathbb{N}$. Let $(x_i)_{i\in\mathbb{N}}$ be a Cauchy sequence in X, with limit x_∞. Define $g\colon \mathbb{N}^+ \to X$ by $g(\alpha) = \lim_{n\to\infty} h(\alpha, n)$ where $h(\alpha, n) = x$ if $\alpha\restriction_n = 1^n$ and $h(\alpha, n) = x_m$ if $\alpha\restriction_n = 1^m0^{n-m}$. By definition, $g(\kappa_\infty) = x_\infty$ and $g(\kappa_i) = x_i$. By 1, $f \circ g$ is sequentially continuous. So, for any $\varepsilon > 0$, there exists m such that, for all $n \geq m$, it holds that $d(f(g(\kappa_n)), f(g(\kappa_\infty))) < \varepsilon$, i.e. $d(f(x_n), f(x_\infty)) < \varepsilon$. Thus f is indeed sequentially continuous. A similar argument establishes that 3 implies $\mathsf{CP}_{\mathsf{seq}}(X, \mathbb{R})$, for all CSMs X. From this 4 follows by using, for any $(x_i)_{i\in\mathbb{N}}$ converging to x in X, and $f\colon X \to Y$, the sequential continuity of the function $x' \mapsto d(f(x), f(x'))\colon X \to \mathbb{R}$, as in the proof of Proposition 4.1. It remains to show that $1 \implies 3$, given $\mathsf{AC}_{1,0}$. We establish $\mathsf{CP}_{\mathsf{seq}}(\mathbb{N}^+, \mathbb{R})$ using the condition of Proposition 4.3.3. Consider any $f\colon \mathbb{N}^+ \to \mathbb{R}$ and $\varepsilon > 0$. Because $\mathbb{N}^+$ is a retract of $\mathbb{N}^\mathbb{N}$, it follows from $\mathsf{AC}_{1,0}$ that there exists $g\colon \mathbb{N}^+ \to \{0, 1\}$ such that $g(\alpha) = 0$ implies $d(f(\alpha), f(\kappa_\infty)) < \varepsilon$, and $g(\alpha) = 1$ implies $d(f(\alpha), f(\kappa_\infty)) > \varepsilon/2$. Thus $g(\kappa_\infty) = 0$ and, by $\mathsf{CP}_{\mathsf{seq}}(\mathbb{N}^+, \mathbb{N})$, there exists m such that, for all $n \geq m$, it holds that $g(\kappa_n) = 0$. Then, for all $n \geq m$, we have $d(f(\kappa_n), f(\kappa_\infty)) < \varepsilon$, as required. □

The next result observes that Propositions 4.1, 4.2 and 4.4 analyse a sequence of successively weaker continuity principles.

Proposition 4.5 $\mathsf{CP}_{\mathsf{pt}}(\mathbb{Z}^\mathbb{N}, \mathbb{N}) \implies \mathsf{CP}_{\mathsf{pt}}(2^\mathbb{N}, \mathbb{N}) \implies \mathsf{CP}_{\mathsf{seq}}(\mathbb{N}^+, \mathbb{N})$.

P r o o f. The first implication holds because $2^\mathbb{N}$ is a retract of $\mathbb{Z}^\mathbb{N}$. The second one follows from Proposition 4.3 and the fact that $\mathbb{N}^+$ is a retract of $2^\mathbb{N}$. □

We now present our application of Theorems 2.4 and 2.5 to continuity principles. In the presence of the sequential continuity principle of Proposition 4.4, the general continuity principles of Propositions 4.1 and 4.2 are implied by many of their instances.

Theorem 4.6 *Suppose that* $\mathsf{CP}_{\mathsf{seq}}(\mathbb{N}^+, \mathbb{N})$ *holds.*

1. *If* X *is an inhabited CSM without isolated points and* $\mathsf{CP}_{\mathsf{pt}}(X, \mathbb{R})$ *holds then so does* $\mathsf{CP}_{\mathsf{pt}}(2^\mathbb{N}, \mathbb{N})$.
2. *If* X *is a locally non-compact inhabited CSM and* $\mathsf{CP}_{\mathsf{pt}}(X, \mathbb{R})$ *holds then so does* $\mathsf{CP}_{\mathsf{pt}}(\mathbb{Z}^\mathbb{N}, \mathbb{N})$.

P r o o f. To prove the first part, consider any function $f : 2^{\mathbb{N}} \to \mathbb{Z}$. By Proposition 4.4, it is sequentially continuous. By Theorem 2.4 there exists $\overline{f} : X \to \mathbb{R}$ such that $f = \overline{f} \circ e$. By assumption, $\overline{f}$ is pointwise continuous, therefore $f = \overline{f} \circ e$ is too. The second part is proved analogously as a consequence of Theorem 2.5. □

One application of the above theorem is to establish the failure of interesting instances of continuity principles, by establishing the failure of $\mathsf{CP}_{\mathsf{pt}}(2^{\mathbb{N}}, \mathbb{N})$ or $\mathsf{CP}_{\mathsf{pt}}(\mathbb{Z}^{\mathbb{N}}, \mathbb{N})$. For example, there is a well-known condition under which $\mathsf{CP}_{\mathsf{pt}}(\mathbb{Z}^{\mathbb{N}}, \mathbb{N})$ fails. We write $\mathsf{AC}_{2,0}$ for the the principle of choice for statements of the form $\forall\, f \in \mathbb{N}^{\mathbb{N}^{\mathbb{N}}} .\, \exists\, n \in \mathbb{N} .\, \varphi$.

Proposition 4.7 *If* $\mathsf{AC}_{2,0}$ *holds then* $\mathsf{CP}_{\mathsf{pt}}(\mathbb{Z}^{\mathbb{N}}, \mathbb{N})$ *does not.*

P r o o f. See Section 9.6.10 of [21]. □

Corollary 4.8 *If both* $\mathsf{CP}_{\mathsf{seq}}(\mathbb{N}^{+}, \mathbb{N})$ *and* $\mathsf{AC}_{2,0}$ *hold then, for any inhabited locally non-compact CSM* X*, the continuity principle* $\mathsf{CP}_{\mathsf{pt}}(X, \mathbb{R})$ *is not true.*

P r o o f. If all functions $X \to \mathbb{R}$ were continuous, then by the second part of Theorem 4.6, $\mathsf{CP}_{\mathsf{pt}}(\mathbb{Z}^{\mathbb{N}}, \mathbb{N})$ would hold, but by Proposition 4.7 this would contradict $\mathsf{AC}_{2,0}$. □

We remark that Proposition 4.7 and Corollary 4.8 rely on the extensionality of functions.

We now step back from the preceding development within constructive mathematics, and survey a few of the familiar and less familiar constructive scenarios in which various of the continuity principles discussed above either hold or fail.

Example 4.9 In Brouwer's intuitionism both $\mathsf{CP}_{\mathsf{pt}}(\mathbb{Z}^{\mathbb{N}}, \mathbb{N})$ and $\mathsf{AC}_{1,0}$ are valid. So the full power of Proposition 4.1.4 is available. This situation is mimicked within the internal logic of the realizability topos $\mathsf{RT}(K_2)$ over Kleene's second algebra K_2 [13, 1].

Example 4.10 In Markov's Recursive Mathematics, $\mathsf{CP}_{\mathsf{pt}}(\mathbb{Z}^{\mathbb{N}}, \mathbb{R})$ is valid (although $\mathsf{AC}_{1,0}$ fails), and hence Proposition 4.1.4 is again available. This situation is mimicked within the internal logic of Hyland's *effective topos*, $\mathcal{E}\!f\!f$, [11].

Example 4.11 In the realizability toposes $\mathsf{RT}(\mathcal{P}\omega)$ and $\mathsf{RT}(D)$, where D is a universal Scott domain, the continuity principle $\mathsf{CP}_{\mathsf{pt}}(2^{\mathbb{N}}, \mathbb{N})$ holds, as a consequence of the existence of a continuous modulus of uniform continuity on Cantor space $2^{\mathbb{N}}$. (In fact a stronger continuity principle holds: all functions from $2^{\mathbb{N}}$ to $\mathbb{N}$ are *uniformly* continuous.) Also choice holds between arbitrary "finite types" [2, 1], in particular $\mathsf{AC}_{1,0}$ and $\mathsf{AC}_{2,0}$ hold. Thus, by Proposition 4.2, $\mathsf{CP}_{\mathsf{pt}}(X, Y)$ holds, for every locally CTB CSM X and metric space Y. On the other hand, by Corollary 4.8, $\mathsf{CP}_{\mathsf{pt}}(X, \mathbb{R})$ fails, for any inhabited locally non-compact CSM X. In particular, $\mathsf{CP}_{\mathsf{pt}}(\mathcal{C}_{\mathsf{u}}[-1, 1], \mathbb{R})$ fails, for, by Proposition 2.3, $\mathcal{C}_{\mathsf{u}}[-1, 1]$ is an inhabited locally non-compact CSM. Thus we have generalized the main result of [7], which established the failure of $\mathsf{CP}_{\mathsf{pt}}(\mathcal{C}_{\mathsf{u}}[-1, 1], \mathbb{R})$ in $\mathsf{RT}(D)$. (Actually, [7], work with the set $\mathcal{C}[-1, 1]$ of pointwise continuous functions from $[-1, 1]$. In $\mathsf{RT}(D)$, it holds that $\mathcal{C}[-1, 1] = \mathcal{C}_{\mathsf{u}}[-1, 1]$, using $\mathsf{AC}_{1,0}$ and the uniform continuity of functions from $2^{\mathbb{N}}$ to $\mathbb{N}$.)

Example 4.12 In the *extensional effective topos* [18, 22], by construction, choice holds for all "finite types"; in particular, $\mathsf{AC}_{1,0}$ and $\mathsf{AC}_{2,0}$ hold. Peter Lietz [14, Chapter II] shows that $\mathsf{CP}_{\mathsf{seq}}(\mathbb{N}^{+}, \mathbb{N})$ holds, but that $\mathsf{CP}_{\mathsf{pt}}(2^{\mathbb{N}}, \mathbb{N})$ fails. By Proposition 4.4, it follows that $\mathsf{CP}_{\mathsf{seq}}(X, Y)$ holds for every CSM X and metric space Y; however, by Theorem 4.6.1, $\mathsf{CP}_{\mathsf{pt}}(X, \mathbb{R})$ fails for any inhabited CSM X without isolated points. In particular, $\mathsf{CP}_{\mathsf{pt}}(\mathbb{R}, \mathbb{R})$ fails. This shows that constructively $\mathsf{CP}_{\mathsf{seq}}(\mathbb{R}, \mathbb{R})$ does not imply $\mathsf{CP}_{\mathsf{pt}}(\mathbb{R}, \mathbb{R})$.

5 An Application to Banach-Mazur computability

In this section, we switch to ordinary classical mathematics.[1] Our aim is to prove a general (classical) result, Theorem 5.2, that differentiates between computability in the sense of Markov, which is the most widely recognised notion of computability, and computability in the sense of Banach and Mazur. To achieve this, we apply

[1] Theorem 5.2, which states the existence of functions that are not Markov computable, is obviously not provable using only constructive principles consistent with Church's Thesis, CT_0 [20, §4.3].

the results of Section 3, making crucial use of their constructivity in order to use them as statements valid in the internal logic of Hyland's *effective topos*, $\mathcal{E}\!f\!f$, [11].

5.1 Numbered sets, Markov computability and computable metric spaces

Following [9, 10], we introduce the notion of Markov and Banach-Mazur computability in the setting of computable metric spaces presented as numbered sets.

A *numbered set*, also known as *modest set*, is a structure $X = (|X|, \nu_X)$ where X is a set, and ν_X is a partial surjection from $\mathbb{N}$ onto X. This is a widely used generalization of Eršov's notion of numbered set. Indeed, an *Eršov numbered set* is just a numbered set X for which ν_X is a total function.

We shall be interested in different notions of morphism between numbered sets X, Y. A *function* $f : X \to Y$ is simply a (set-theoretic) function $f : |X| \to |Y|$. A function $f : X \to Y$ is said to be *Markov computable* (henceforth simply *computable*) if there exists a partial-recursive function $r : \mathbb{N} \rightharpoonup \mathbb{N}$ such that $f \circ \nu_X = \nu_Y \circ r$; in this situation we say that r *realizes* f.

The category of numbered sets and computable functions is cartesian closed. Finite products are easily defined using a pairing function $(-,-)$ on natural numbers. The function space Y^X has the set of computable functions from X to Y as its underlying set, and ν_{Y^X} is the unique partial surjection for which $\nu^{-1}_{Y^X}(f)$ is the set of indices (in some standard enumeration) of all partial recursive functions realizing f. Furthermore, the numbered set $\mathsf{N} = (\mathbb{N}, \mathsf{id}_{\mathbb{N}})$ is a natural numbers object in the category. We write 2 for the set $\{0, 1\}$ numbered by the (partial) identity. And we write Z for $\mathbb{Z}$ numbered by a computable bijection from $\mathbb{N}$ to $\mathbb{Z}$.

Numbered sets come with an associated intuitionistic logic for reasoning about them, derived from their embedding within Hyland's *effective topos*, $\mathcal{E}\!f\!f$, [11]. Indeed, the category of numbered sets and computable functions between embeds fully as the category of *effective objects*, or *modest sets*, in $\mathcal{E}\!f\!f$, see [11, Section 7]. The induced logic allows a theory of computable metric spaces (and other aspects of constructive mathematics) to be developed in an entirely routine way, by merely interpreting the standard constructive definitions within the internal logic of the topos. Although, in effect, this is the route we now follow, we shall present all definitions in concrete form, in order to make them accessible to readers who are not familiar with the effective topos. At the same time, we also state the equivalent logical definitions, in order to avoid making the paper unnecessarily impenetrable to those who *are* familiar with the topos-theoretic approach.

The computable real numbers are defined as a numbered set $\mathsf{R_c}$ in a standard way, see e.g. [9]. A *computable metric space* is given by a numbered set X together with a computable distance function $d : X \times X \to \mathsf{R_c}$, satisfying the usual axioms. A *(computable) Cauchy sequence* in a computable metric space is given by a computable sequence, i.e. a computable function $x_{(-)} : \mathsf{N} \to X$ for which there exists a computable *modulus* function $\mu : \mathsf{N} \to \mathsf{N}$ satisfying $d(x_i, x_j) < 2^{-n}$ for all n and $i, j \geq \mu(n)$. The *limit*, if it exists, of a Cauchy sequence $x_{(-)}$ with modulus μ, is the unique element $x \in X$ satisfying $d(x_i, x) \leq 2^{-n}$ for all n and $i \geq \mu(n)$. The numbered set $\mathrm{Cauchy}(X)$ of Cauchy sequences in X is defined by setting $\nu^{-1}(x_{(-)})$ to be the set of all pairs (e, e') where e is an index for the sequence $x_{(-)}$ and e' is an index for a modulus μ. A computable metric space is said to be *complete* if every Cauchy sequence has a limit, and the limit-finding function $\mathrm{Cauchy}(X) \to X$ is computable. It is said to be *separable* if there exists a computable sequence $s_{(-)}$ in X and there exists a partial recursive function $c : \mathbb{N} \times \mathbb{N} \rightharpoonup \mathbb{N}$ such that, for all $e \in \mathrm{dom}(\nu_X)$ and $n \in \mathbb{N}$, it holds that $c(e, n)$ is defined and $d(\nu_X(e), s_{c(e,n)}) < 2^{-n}$. A *computable CSM* is a complete and separable computable metric space.

The above definitions arise naturally in the context of the effective topos. The object $\mathsf{R_c}$ is just the object of Cauchy (equivalently Dedekind) reals in $\mathcal{E}\!f\!f$. A numbered set with computable distance function is a computable metric space if, and only if, the corresponding effective object and distance function in $\mathcal{E}\!f\!f$ form a metric space in the internal logic of $\mathcal{E}\!f\!f$. Furthermore, the numbered set is a computable CSM if, and only if, the effective object is internally a CSM. These facts are simply consequences of the explicit definitions for computable metric spaces above being direct unwindings of the corresponding internal definitions. Incidentally, it even holds that the separable metric spaces in $\mathcal{E}\!f\!f$ are (up to isomorphism) exactly the computable separable metric spaces as defined above. This is so because every separable metric space is a double-negation separated subquotient of N^{N}, which is in turn a subquotient of N, and the effective objects are (up to isomorphism) just the double-negation separated subquotients of N.

A computable metric space X is said to be *without isolated points* if there exists a partial recursive function $p : \mathbb{N} \times \mathbb{N} \rightharpoonup \mathbb{N}$ satisfying, for all $e \in \mathrm{dom}(\nu_X)$ and $n \in \mathbb{N}$, it holds that $p(e, n) \in \mathrm{dom}(\nu_X)$ and $0 <$

$d(\nu_X(e), \nu_X(p(e,n))) < 2^{-n}$. A computable sequence $x_{(-)}$ in a computable metric space X is said to be *without accumulation point* if there exist partial recursive functions $p : \mathbb{N} \rightharpoonup \mathbb{N}$ and $q : \mathbb{N} \rightharpoonup \mathbb{N}$ satisfying, for all $e \in \mathrm{dom}(\nu_X)$, it holds that $p(e)$ and $q(e)$ are defined and $d(\nu_X(e), x_m) > 2^{-q(e)}$ for all $m \geq p(e)$. A *witness* for a computable sequence without accumulation point is given by a triple (e_1, e_2, e_3) where e_1 is an index for the recursive function realizing $x_{(-)}$, and e_2, e_3 are indices for the partial recursive functions realizing p and q respectively. A computable metric space X is said to be *locally non-compact* if there is a partial recursive function $r : \mathbb{N} \times \mathbb{N} \rightharpoonup \mathbb{N}$ such that, for all $e \in \mathrm{dom}(\nu_X)$ and n it holds that $r(e,n)$ is defined and equal to (e_1, e_2, e_3) where this triple witnesses that a (thereby determined) sequence $x_{(-)}$ is without accumulation point, and where, furthermore, for all $i \in \mathbb{N}$ it holds that $d(\nu_X(e), x_i) < 2^{-n}$.

Once again, the above definitions are simple unwindings of the corresponding internal definitions in $\mathcal{E}\!f\!f$. Thus a computable metric space is without isolated points (respectively locally non-compact) if and only if the corresponding effective object is internally without isolated points (respectively locally non-compact) according to the definitions in Section 2. Trivially the computable CSM of computable real numbers, $\mathsf{R_c}$, is without isolated points. Moreover, because Church's Thesis CT_0 holds in $\mathcal{E}\!f\!f$, see [11], it follows from the remark in Section 2 that $\mathsf{R_c}$ is also locally non-compact.

Question 5.1 *Is every computable CSM without isolated points locally non-compact?*

5.2 Banach-Mazur computability

The main result of this section is concerned with a second notion of computable function between numbered sets, due to Banach and Mazur. A function $f : X \to Y$, is said to be *Banach-Mazur computable*, henceforth *BM-computable*, if, for every computable sequence $s : \mathsf{N} \to X$, it holds that the sequence $f \circ s : \mathsf{N} \to Y$ is computable.

It is obvious that every computable function is BM-computable. Conversely, if X is an Eršov numbered set then it is easily seen that every BM-computable function is computable (and this result generalizes to any X isomorphic to an Eršov numbered set by way of computable isomorphisms). However, many of the interesting objects of computable analysis are not isomorphic to Eršov numbered sets. Although, for such spaces, there is no reason for all BM-computable functions to be computable, it is not easy to find counterexamples. A first (and sophisticated) such example was produced by Friedberg, who showed that there exists a BM-computable function from N^{N} to N that is not computable [8]. A general discussion of the relationship between the two notions is contained in Hertling's recent paper [10], where, as the main result, a BM-computable but non-computable function from $\mathsf{R_c}$ to $\mathsf{R_c}$ is constructed. The main theorem of this section generalizes Hertling's result to a wide class of computable metric spaces.

Theorem 5.2 *If X is an inhabited computable CSM without isolated points then there exists a BM-computable function from X to $\mathsf{R_c}$ that is not computable.*

The proof of Theorem 5.2 is given in Section 5.3 below. An interesting feature of the proof is that we obtain our result as a direct consequence of Friedberg's [8], whereas Hertling's non-computable but BM-computable function from $\mathsf{R_c}$ to $\mathsf{R_c}$ was constructed from first principles [10].

In preparation for the proof, we present three propositions asserting positive properties of BM-computable functions between numbered sets and computable metric spaces. The first states that the object X^{N}, which is an exponential in the category of numbered sets and computable maps, is also an exponential in the category of BM-computable maps (thus the object N is exponentiable in the category of BM-computable maps).

Proposition 5.3 *Suppose that X, Y are numbered sets.*

1. *If $f : \mathsf{N} \times X \to Y$ is BM-computable, then its transpose $\tilde{f} : x \mapsto \lambda n.\, f(n,x)$ is a BM-computable function from X to Y^{N}.*
2. *Conversely, if $g : X \to Y^{\mathsf{N}}$ is BM-computable, then so is the function $\overline{g} : (n,x) \mapsto g(x)(n) : \mathsf{N} \times X \to Y$.*

P r o o f. For statement 1, suppose that $f : \mathsf{N} \times X \to Y$ is BM-computable. We must first verify that, for any $x \in |X|$, it holds that $\tilde{f}(x) \in |Y^{\mathsf{N}}|$, i.e., that $n \mapsto f(n,x) : \mathsf{N} \to Y$ is computable. But $s : n \mapsto (n,x) : \mathsf{N} \to \mathsf{N} \times X$ is a computable sequence, so by BM-computability $f \circ s$ is computable, i.e. $n \mapsto f(n,x)$ is indeed computable. To show that $\tilde{f}$ is BM-computable, consider any computable $x_{(-)} : \mathsf{N} \to X$. Then

$(n,m) \mapsto (n,x_m) : \mathsf{N} \times \mathsf{N} \to \mathsf{N} \times X$ is computable. By BM-computability of f, it holds that $(n,m) \mapsto f(n,x_m) : \mathsf{N} \times \mathsf{N} \to Y$ is computable. Thus, by cartesian closure, $m \mapsto \lambda n.\, f(n,x_m) : \mathsf{N} \to Y^{\mathbb{N}}$ is computable, i.e. $m \mapsto \overline{f}(x_m)$ is computable, as required.

We omit the similar proof of statement 2. (Anyway, the result is not used below.) □

Mazur [15] proved that every BM-computable function from $\mathsf{R_c}$ to $\mathsf{R_c}$ enjoys the property that, for any computable Cauchy sequence $(x_n)_n$ with limit x_∞, it holds that $(f(x_n))_n$, considered as a sequence of ordinary (though, of course, computable) real numbers, is a Cauchy sequence *in the ordinary sense* with limit $f(x_\infty)$. The next proposition is an improvement on this result due to Hertling, [9]. The improvement both generalizes Mazur's result to the setting of computable metric spaces, and also strengthens it to show that the derived sequence $(f(x_n))_n$ is even a computable Cauchy sequence in the computable sense.

Proposition 5.4 (Hertling) *If X, Y are computable metric spaces, with X complete, and $f : X \to Y$ is BM-computable then, for any computable Cauchy sequence $(x_n)_n$ with limit x_∞, it holds that $(f(x_n))_n$ is a computable Cauchy sequence with limit $f(x_\infty)$.*

P r o o f. This is proved as Theorem 17 of [9], where, in fact, the proof is given for a more general notion of "BM-computable metric space". □

The next proposition concerns the numbered set N^+, representing the one-point compactification of N in $\mathcal{E}\!f\!f$, whose underlying set is $\{\kappa_n \mid n \in \mathbb{N}\} \cup \{\kappa_\infty\}$, where, for $i \in \mathbb{N} \cup \{\infty\}$, the sequence $\kappa_i \in 2^{\mathbb{N}}$ satisfies $\kappa_i(n) = 1$ if and only if $n < i$. The set $\nu^{-1}(\kappa_i)$ is inherited from 2^{N}.

Proposition 5.5 *If X is a computable complete metric space then every BM-computable function from N^+ to X is computable.*

P r o o f. Suppose $g : \mathsf{N}^+ \to X$ is BM-computable. The sequence $(\kappa_n)_n$ in N^+ is computable, and also a Cauchy sequence with limit κ_∞. As g is BM-computable, the sequence $(g(\kappa_n))_n$ is computable. By Proposition 5.4, it is Cauchy with limit $g(\kappa_\infty)$.

We show below that there is a partial-recursive function $p : \mathbb{N} \times \mathbb{N} \rightharpoonup \mathbb{N}$ satisfying: (i) for all $\alpha \in \mathsf{N}^+$ and (n,m) with $m \in \nu_{\mathsf{N}^+}^{-1}(\alpha)$, it holds that $d(\nu_X(p(n,m)), g(\alpha)) \leq 2^{-n}$; and (ii) $m, m' \in \nu_{\mathsf{N}^+}^{-1}(\alpha)$ implies $\nu_X(p(n,m)) = \nu_X(p(n,m'))$. Given such a p, we have that p realizes a computable function $h : \mathsf{N} \times \mathsf{N}^+ \to X$, whose transpose $\tilde{h} : \mathsf{N}^+ \to X^N$ maps every α in N^+ to a Cauchy sequence in X with limit $g(\alpha)$. Because limits of Cauchy sequences are found computably, it follows that g is computable.

It remains to define p. This is given by the following algorithm. Given (n,m), first compute $k = \mu(n)$, where μ is the computable modulus function for the sequence $(g(\kappa_i))_i$ (that is, for all $j, j' \geq \mu(n)$, it holds that $d(g(\kappa_j), g(\kappa_{j'})) \leq 2^{-n}$). Next, examine the values $\{m\}(0), \ldots, \{m\}(k)$. If any of these values is undefined, or ≥ 2, or if a 0 occurs before a 1 then m cannot be in any set $\nu_{\mathsf{N}^+}^{-1}(\alpha)$, and $p(n,m)$ is left undefined. Otherwise, let i be the smallest number with $0 \leq i \leq k$ such that $\{m\}(i) = 0$, if such an i exists, or let i be k, if $\{m\}(0), \ldots, \{m\}(k)$ are all 1. Finally, define $p(n,m)$ to be the element of $\nu_X^{-1}(g(\kappa_i))$, which can be computed using the realizer of the computable sequence $(g(\kappa_n))_n$.

It is immediate from the definition that p satisfies property (ii). We must show that it also satisfies (i). Suppose then that $\alpha \in \mathsf{N}^+$ and $m \in \nu_{\mathsf{N}^+}^{-1}(\alpha)$. Then, for any n, define $k = \mu(n)$ as above. There are three cases. First, if $\alpha = \kappa_j$ for some $j \leq k$, then the i defined above is j and so $\kappa_i = \alpha$. It follows that $p(n,m) \in \nu_X^{-1}(g(\alpha))$, so $d(\nu_X(p(n,m)), g(\alpha)) = 0 < 2^{-n}$. Second, if $\alpha = \kappa_j$, for some $j > k$, then the i defined above is k. So $d(\nu_X(p(n,m)), g(\alpha)) = d(\kappa_k, \kappa_j) \leq 2^{-n}$, by the modulus property of $k = \mu(n)$. Third, if $\alpha = \kappa_\infty$ then i is again k, so $d(\nu_X(p(n,m)), g(\alpha)) = d(\kappa_k, \kappa_\infty) \leq 2^{-n}$, by the definition of limit and the modulus property of $k = \mu(n)$. □

5.3 Proof of Theorem 5.2

Our proof of Theorem 5.2 applies the results of Section 3 within the context of the effective topos, $\mathcal{E}\!f\!f$. In order to fully understand the proof as written, it is necessary for the reader to have some knowledge of the workings of the internal logic of $\mathcal{E}\!f\!f$. However, we also provide concrete descriptions of the numbered sets involved in the proof, so that a reader with no knowledge of $\mathcal{E}\!f\!f$ should nevertheless be able to fill in the various recursion-theoretic details that are otherwise taken care of automatically by the internal logic of $\mathcal{E}\!f\!f$.

The main additional tool we need in the proof is the following adaptation of Proposition 3.2 to a statement about BM-computable functions between numbered sets. Let $i : \mathsf{Z} \to \mathsf{R_c}$ be the inclusion of the numbered set Z in the computable reals.

Lemma 5.6 *For every BM-computable $f : \mathsf{Z}^{\mathsf{N}} \to \mathsf{Z}$ there exists a BM-computable $h : {\mathsf{R_c}}^{\mathsf{N}} \to \mathsf{R_c}$ satisfying $h \circ i^{\mathsf{N}} = i \circ f : \mathsf{Z}^{\mathsf{N}} \to \mathsf{R_c}$.*

Before giving the proof, we show that Theorem 5.2 is indeed a consequence of the lemma.

Suppose that X is an inhabited computable CSM without isolated points. Thus X is an effective object of $\mathcal{E}\!f\!f$, which is a CSM without isolated points in the internal logic of $\mathcal{E}\!f\!f$. It is now an immediate consequence of Proposition 3.22 that there exist computable functions $e : 2^{\mathsf{N}} \to X$ and $g : X \to {\mathsf{R_c}}^{\mathsf{N}}$ such that $g \circ e = j^{\mathsf{N}} : 2^{\mathsf{N}} \to {\mathsf{R_c}}^{\mathsf{N}}$, where $j : 2 \to \mathsf{R_c}$ is the inclusion of the numbered set $\{0, 1\}$ in the computable reals.

By Friedberg's theorem [8], there exists a BM-computable function from N^{N} to N that is not computable. Moreover, the numbered sets 2^{N} and N^{N} are computably isomorphic (indeed pointwise homeomorphic) [3, IV.13]. Therefore there exists a BM-computable $f_F : 2^{\mathsf{N}} \to \mathsf{N}$ that is not computable. By Lemma 5.6, using that 2 is a computable retract of Z, there exists a BM-computable $h_F : {\mathsf{R_c}}^{\mathsf{N}} \to \mathsf{R_c}$ such that $h_F \circ j^{\mathsf{N}} = i \circ f_F : 2^{\mathsf{N}} \to \mathsf{R_c}$. Thus $i \circ f_F = h_F \circ g \circ e$. Because g and e are computable and i is the inclusion of Z in $\mathsf{R_c}$, it follows that if $h_F \circ g$ were computable then f_F would be too, which is not the case. Thus $h_F \circ g : X \to \mathsf{R_c}$ is indeed BM-computable but not computable. This completes the proof of Theorem 5.2, given Lemma 5.6.

It remains to prove Lemma 5.6. This cannot be derived directly by interpreting Proposition 3.2 in $\mathcal{E}\!f\!f$, because BM-computable functions only live inside $\mathcal{E}\!f\!f$ when they happen to be computable. Instead, we construct h using the extension property for adequate subdomain pairs, as defined at the end of Section 3, using Lemmas 3.20 and 3.21 to show that the definition has the required properties. In order to effect the required construction, we need to make use of the lemma below.

Lemma 5.7 *Every BM-computable function from $\mathsf{N} \times \mathsf{N}^+$ to N is computable.*

P r o o f. Let $f : \mathsf{N} \times \mathsf{N}^+ \to \mathsf{N}$ be BM-computable. By Proposition 5.3.1, the transpose $\tilde{f} : \mathsf{N}^+ \to \mathsf{N}^{\mathsf{N}}$ is BM-computable, and hence, by Proposition 5.5, computable. Thus, by the cartesian closure of the category of numbered sets and computable maps, f is indeed computable. □

We now prove Lemma 5.6. Henceforth, let $f : \mathsf{Z}^{\mathsf{N}} \to \mathsf{Z}$ be any BM-computable function. We must define the $h : {\mathsf{R_c}}^{\mathsf{N}} \to \mathsf{R_c}$ required by Lemma 5.6. First, we define h as a function.

Given any $\gamma \in {\mathsf{R_c}}^{\mathsf{N}}$, let $\beta \in \mathsf{Z}^{\mathsf{N}}$ be adequate for γ. Such a computable β exists, because the existence of β is true in $\mathcal{E}\!f\!f$. Define B_β to be the subobject of Z^{N} defined in $\mathcal{E}\!f\!f$ by:

$$B_\beta = \{\alpha \in \mathsf{Z}^{\mathsf{N}} \mid \forall\, n \in \mathsf{N}\,.\,(\alpha_n \neq \beta_n \implies \forall\, m \geq n\,.\,\alpha_n = 0)\}\,. \tag{7}$$

(As a numbered set, B_β is given concretely as the evident subset of Z^{N} with numbering defined by $\nu_{B_\beta}^{-1}(\alpha) = \nu_{\mathsf{Z}^{\mathsf{N}}}^{-1}(\alpha)$.) We write $\iota : B_\beta \to \mathsf{Z}^{\mathsf{N}}$ for the inclusion.

We now define maps $r_\beta : \mathsf{N}^+ \to B_\beta$ and $s_\beta : B_\beta \to \mathsf{N}^+$ as follows:

$$r_\beta(\alpha) = \lambda n\!:\!\mathbb{N}\,.\begin{cases}\beta(n) & \text{if } \forall\, m \leq n\,.\,\alpha(m) = 1,\\ 0 & \text{otherwise,}\end{cases}$$

$$s_\beta(\alpha) = \lambda n\!:\!\mathbb{N}\,.\begin{cases}1 & \text{if } \forall\, m \leq n\,.\,\alpha(m) = \beta(m),\\ 0 & \text{otherwise.}\end{cases}$$

It is easily seen that $r_\beta \circ s_\beta$ is the identity on B_β. Thus B_β is a computable retract of N^+ and hence of $\mathsf{N} \times \mathsf{N}^+$. So, by Lemma 5.7, every BM-computable function from B_β to $\mathsf{Z} \cong \mathsf{N}$ is computable.

Define G_γ to be the singleton subobject of ${\mathsf{R_c}}^{\mathsf{N}}$ containing γ. By (7), it holds that G_γ, B_β together form an adequate subdomain pair in the sense of Section 3.2. Also, the function $f \circ \iota : B_\beta \to \mathsf{Z}$ is BM-computable hence, as observed above, computable. Thus $f \circ \iota$ is a morphism in $\mathcal{E}\!f\!f$. As $\mathsf{CP_{pt}}(\mathbb{N}^+, \mathbb{N})$ holds in $\mathcal{E}\!f\!f$, see [11], $\mathsf{CP_{seq}}(\mathbb{N}^+, \mathbb{N})$ holds as well by Proposition 4.3. Since B_β as a subspace of Z^{N} is a CSM by (7), it follows from Proposition 4.4 that $f \circ \iota$ is sequentially continuous in the internal logic of $\mathcal{E}\!f\!f$. Therefore, the construction of Section 3.2 produces $h_{(G_\gamma, B_\beta, f \circ \iota)} : G_\gamma \to \mathsf{R_c}$. Define $h(\gamma) = h_{(G_\gamma, B_\beta, f \circ \iota)}(\gamma)$.

We have now defined a function $h : \mathsf{R_c}^{\mathsf{N}} \to \mathsf{R_c}$. It is immediate from Lemma 3.21 that $h \circ i^{\mathsf{N}} = i \circ f : \mathsf{Z}^{\mathsf{N}} \to \mathsf{R_c}$. To complete the proof of Lemma 5.6, it remains to be shown that h is BM-computable.

Let $(\gamma_n)_n : \mathsf{N} \to \mathsf{R_c}^{\mathsf{N}}$ be a computable sequence. We must show that the sequence $(h(\gamma_i))_i : \mathsf{N} \to \mathsf{R_c}$ is computable. Let $(\beta_n)_n : \mathsf{N} \to \mathsf{Z}^{\mathsf{N}}$ be a computable sequence such that each β_n is adequate for γ_n. The existence of $(\beta_n)_n$ is easily shown using countable choice in the internal logic of $\mathcal{E}\!f\!f$. Define $B_{(\beta_n)_n}$ to be the subobject of Z^{N} defined in $\mathcal{E}\!f\!f$ by:

$$B_{(\beta_n)_n} = \{\alpha \in \mathsf{Z}^{\mathsf{N}} \mid \exists\, n \in \mathsf{N} \,.\, \alpha \in B_{\beta_n}\} \;.$$

(Concretely, $B_{(\beta_n)_n}$ is the evident subset of Z^{N} with numbering defined by

$$\nu^{-1}_{B_{(\beta_n)_n}}(\alpha) = \{(n,m) \mid \alpha \in |B_{\beta_n}| \wedge m \in \nu^{-1}_{B_{\beta_n}}(\alpha)\} \;,$$

for $\alpha \in B_{(\beta_n)_n}$.) Let $\varepsilon : B_{(\beta_n)_n} \to \mathsf{Z}^{\mathsf{N}}$ be the inclusion.

Define $\rho : \mathsf{N} \times \mathsf{N}^+ \to B_{(\beta_n)_n}$ by

$$\rho(n, \alpha) = r_{\beta_n}(\alpha) \;,$$

making use of the retraction $r_{\beta_n} : \mathsf{N}^+ \to B_{\beta_n}$ defined above. Reasoning internally in $\mathcal{E}\!f\!f$, consider any $\alpha \in B_{(\beta_n)_n}$. Then there exists n such that $\alpha \in B_{\beta_n}$. So $\alpha = r_{\beta_n}(s_{\beta_n}(\alpha)) = \rho(n, s_{\beta_n}(\alpha))$. Thus there exists $(n, \alpha') \in \mathsf{N} \times \mathsf{N}^+$ such that $\alpha = \rho(n, \alpha')$. This shows that the function ρ is epi in $\mathcal{E}\!f\!f$. It follows that, for any numbered set Z and function $u : |B_{(\beta_n)_n}| \to |Z|$, if $u \circ \rho$ is computable then so is u. We have that $f : \mathsf{Z}^{\mathsf{N}} \to \mathsf{Z}$ is BM-computable. Thus $f \circ \varepsilon \circ \rho : \mathsf{N} \times \mathsf{N}^+ \to \mathsf{Z}$ is BM-computable, hence, by Lemma 5.7, computable. Therefore $f \circ \varepsilon : B_{(\beta_n)_n} \to \mathsf{Z}$ is computable.

Let $G_{(\gamma_n)_n}$ be the subobject of $\mathsf{R_c}^{\mathsf{N}}$ defined by:

$$G_{(\gamma_n)_n} = \{\gamma \in \mathsf{R_c}^{\mathsf{N}} \mid \exists\, n \in \mathsf{N} \,.\, \gamma = \gamma_n\} \;.$$

(As a numbered set, this has the obvious underlying set, and the numbering can be taken to be $\nu^{-1}_{G_{(\gamma_n)_n}}(\gamma) = \{n \mid \gamma = \gamma_n\}$.) By these definitions, it follows that $G_{(\gamma_n)_n}, B_{(\beta_n)_n}$ form an adequate subdomain pair. As $f \circ \varepsilon : B_{(\beta_n)_n} \to \mathsf{Z}$ is a morphism in $\mathcal{E}\!f\!f$ and thus sequentially continuous, the construction of Section 3.2 produces a computable $h_{(G_{(\gamma_n)_n}, B_{(\beta_n)_n}, f\circ\varepsilon)} : G_{(\gamma_n)_n} \to \mathsf{R_c}$. Moreover, by Lemma 3.21 and the definition of h, for any $\gamma \in G_{(\gamma_n)_n}$, it holds that $h_{(G_{(\gamma_n)_n}, B_{(\beta_n)_n}, f\circ\varepsilon)}(\gamma) = h(\gamma)$. Thus the total recursive function showing that $h_{(G_{(\gamma_n)_n}, B_{(\beta_n)_n}, f\circ\varepsilon)}$ is computable witnesses the computability of the sequence $(h(\gamma_i))_i : \mathsf{N} \to \mathsf{R_c}$. This shows that h is indeed BM-computable, and so concludes the proof of Lemma 5.6.

Remark 5.8 The above combination of internal and external reasoning is essential to our proof because the BM-computable functions do not live inside $\mathcal{E}\!f\!f$. An interesting alternative would be to instead apply Theorem 2.4 directly in the context of Mulry's "recursive topos" [16], in which the morphisms (between certain objects) are exactly the BM-computable functions. Such an approach may be possible, but it is non-trivial because the internal logic of Mulry's topos is awkward to use; for example, it is necessary to work with a non-standard object of natural numbers, for which only restricted forms of induction are available, see [19]. Indeed, we do not know whether the proof of Theorem 2.4 goes through in this setting.

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