On Isomorphism Invariance and Isomorphism Reflection in Type Theory

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Thank you for the invitation, I am honored to be here. Most of the results presented here are joint work with Théo Winterhalter. We first did this work when he visited Ljubljana some years ago as a Master's student, and you can read about it in his PhD dissertation.

Outline

Isomorphism Invariance

- Overview

Isomorphism Reflection

- Consistency & models
- So what?

I would like to speak about a principle that pervades mathematical practice but often eludes formalization.

As a warmup, we shall review the principle of Isomorphism invariance.

We shall then focus on a special case, the Isomorphism reflection (I shall explain what it is).
The principle is not obviously consistent, so it will be good to have some models.
We shall also ask whether there the principle can be put to good use.

Isomorphism invariance

If $A \cong B$ and $\Phi(A)$ then $\Phi(B)$

To set the stage, let us recall an important principle in mathematics.

Isomorphism invariance is the principle stating, roughly, that whatever we can do with a structure we can do with one that is isomorphic to it.

You can read Φ as a logical formula, but since this is TYPES we should read it as a type constructor.

Principle of Isomorphism

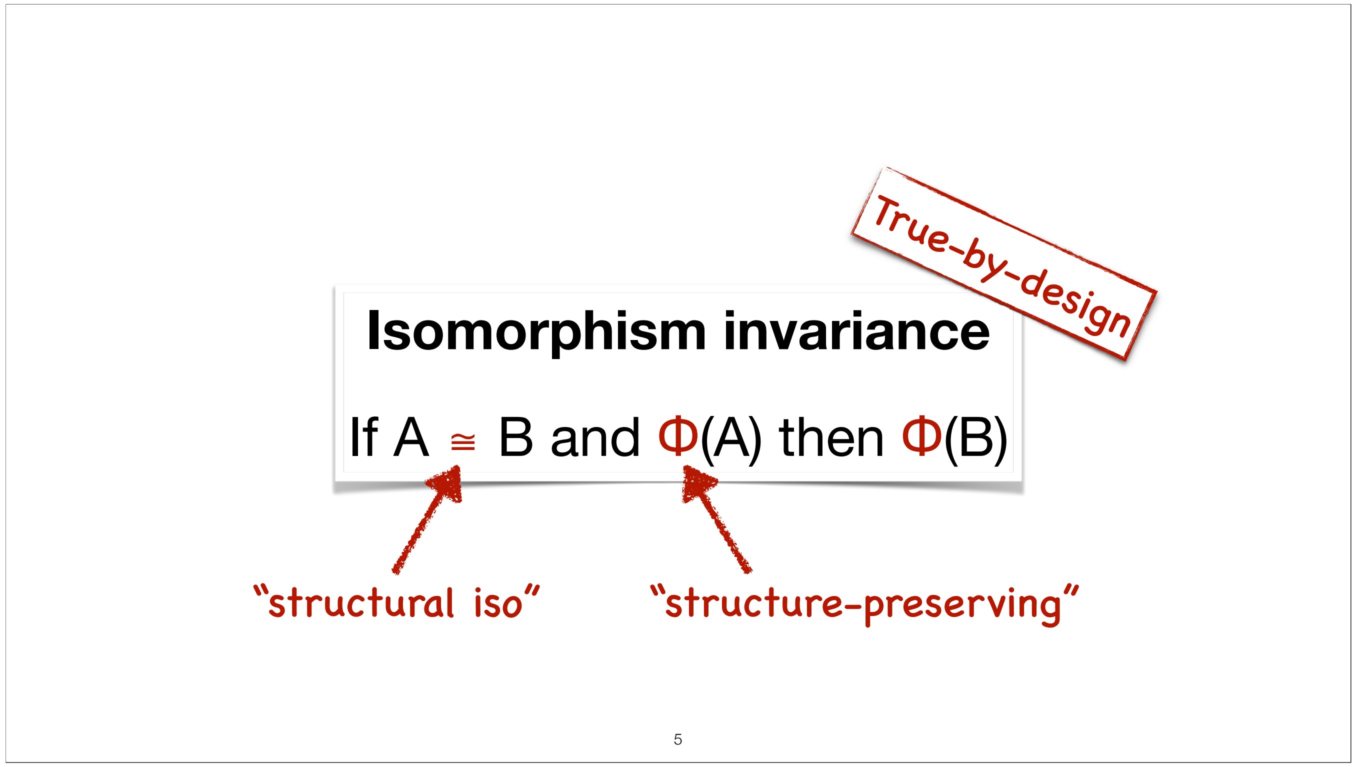
Principle of Structuralism

Isomorphism invariance

If $A \cong B$ and $\Phi(A)$ then $\Phi(B)$

The principle has also been called the **Principle of Isomorphism** by Michael Makkai, and the **Principle of Structuralism** by Steve Awodey.

The version shown here is informal, and there are several ways of understanding it.



For example, we can make it a truism by stating that it only applies to **structure-preserving** Φ,
which immediately begs the question: what is a structure-preserving statement or construction, if not one that is invariant under isomorphism?

Informal explanations often take this route. As a student I was told that a “topological property” on a space that is one that is preserved under homeomorphism, and that topology is the study of topological invariants, and similarly for other branches of mathematics. For example, if A and B are categories, we would want to read ≅ as equivalence of categories.

If one works in a setting where non-invariant constructions are allowed (such as first-order logic and set theory), then one probably has to save the principle this way.

In the context of category theory Michael Makkai worked out FOLDS – first-order logic over dependent sorts – which guarantee isomorphism invariance. (Caveat: FOLDS do not have a primitive =.)

How about the isomorphism invariance in type theory?

Meta-theoretic
Isomorphism invariance

If $\vdash e : A \cong B$ then $\vdash e' : \Phi(A) \cong \Phi(B)$ for some e'.

One option is to read the statement **meta-theoretically**, so that it says something about definable constructions: whenever the type of isomorphisms $A \cong B$ is inhabited, for well-formed type expressions A and B, then also $\Phi(A) \cong \Phi(B)$ is inhabited. Here $\Phi(X)$ is a type expression definable from a type metavariable X.

Meta-theoretic
Isomorphism invariance

If $\vdash e : A \cong B$ then $\vdash e' : \Phi(A) \cong \Phi(B)$ for some e'.

- **Simple type theory**: A. Lindenbaum & A. Tarski, *Über die Beschränktheit der Ausdrucksmittel deduktiver Theorien*, Ergebnisse eines mathematischen Kolloquiums, Heft 7 (für das Jahr 1934-35, erschienen 1936), pp.15–22.

This property was established for simple type theory in the 1930’s by Lindenbaum and Tarski.

Meta-theoretic Isomorphism invariance

If $\vdash e : A \cong B$ then $\vdash e' : \Phi(A) \cong \Phi(B)$ for some e'.

- **Per Martin-Löf:** *Invariance Under Isomorphism and Definability,* Ernest Nagel Lectures in Philosophy & Science, Carnegie Mellon University, March 2013
- **"SOGAT":** Rafaël Bocquet, *External Univalence For Second-Order Generalized Algebraic Theories*, November 2022, arxiv.org/abs/2211.07487

For dependent type theories, I had to ask on the TYPES mailing list, and I got two answers.

In 2013 Per Martin-Löf gave a series of lectures in which he establishes isomorphism invariance for MLTT. The lectures are recorded on YouTube. You may take one guess which type theory he considered.

A more modern and general treatment was established by Rafaël Bocquet quite recently.

Both approaches use the same essential idea based on relational models of type theory.

Isomorphism reflection

Let us move to the interesting part, isomorphism reflection.

Isomorphism invariance

If $A \cong B$ and $\Phi(A)$ then $\Phi(B)$

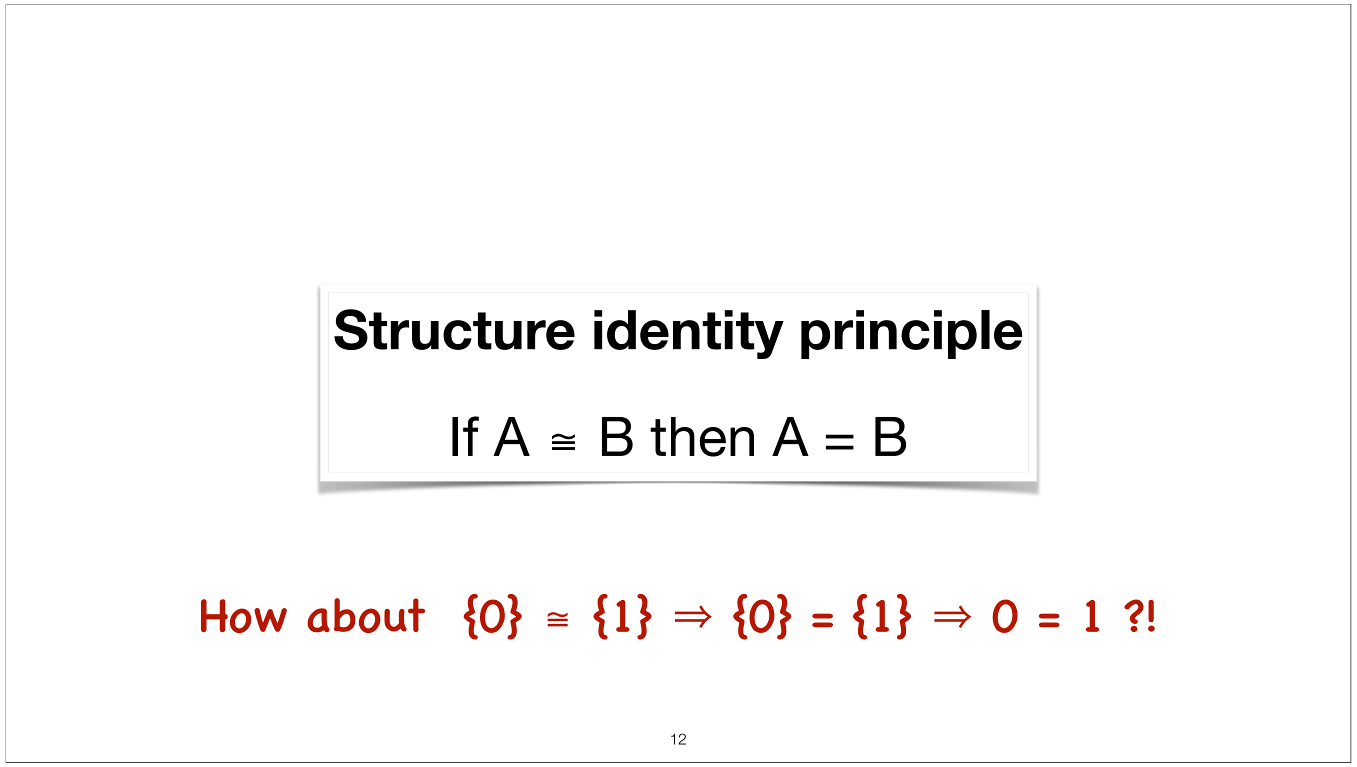
take $\Phi(X) := (A = X)$

There is a critical case of isomorphism invariance, which arises when we take Φ(X) to be A = X. (What = means still remains to be discussed, we're doing things informally right now.)
Since A = A holds, the implication simplifies, as follows.

Structure identity principle

If A ≅ B then A = B

We might call this the **Structure identity principle**: isomorphic structures are identical.



We might call this the **Structure identity principle**: isomorphic structures are identical.

The working mathematician has a love-hate relationship with it. On one hand they use it all the time in informal practice. On the other, they cannot bring themselves to formally declare the principle true. Taken literally, the principle seemingly implies all kinds of nonsense. For instance, $\{0\} \cong \{1\}$, therefore $\{0\} = \{1\}$ and $0 = 1$?!

Structure identity principle

If A ≅ B then A = B

Isomorphism invariance

If A ≅ B and Φ(A) then Φ(B)

The principle is just as strong as Isomorphism invariance:
If A ≅ B and Φ(A) then ϕ(B) follows because A = B.

Let us look at the status of the Structure identity principle in various settings.

Structure identity principle
If $A \cong B$ then $A = B$

Set theory I

Working in *material* set theory such as ZFC,
if $A \cong B$ is existence of a bijection $b : A \to B$,
the principle is false.

If $\cong$ is interpreted as existence of bijection, then material set theory falsifies the principle.

Actually, it validates it only in one case, $A = B = \varnothing$.

Structure identity principle
If $A \cong B$ then $A = B$

Set theory II

Working in *material* set theory such as ZFC,
if $A \cong B$ is equivalence of $\in$-structures,
this is the set-theoretic *Extensionality axiom:*

$$(\forall x . x \in A \Leftrightarrow x \in B) \Rightarrow A = B$$

Still working in material set theory, if we read $\cong$ as isomorphism of sets qua $\in$-structures, we obtain the set-theoretic extensionality axiom.

Structure identity principle
If $A \cong B$ then $A = B$

Type theory I – Univalence

Reading the statement *internally* as

$$\mathrm{Equiv}(A, B) \to \mathrm{Id}_U(A, B)$$

follows from the **univalence axiom**.

Can we internalize the statement? Vladimir Voevodsky gave an answer: read ≅ as equivalence (rather than isomorphism) and = as the identity type. The principle we get is not quite univalence, but it follows from it easily.

This move has shown to be very successful, of course. Yes, it requires a serious commitment by its practitioners, as it makes equality proof relevant.

But what lies at the other side of the spectrum? What if we insist that = be proof irrelevant? In type theory that would mean taking judgmental equality in place of the identity type.

Structure identity principle
If $A \cong B$ then $A = B$

Type theory II – Isomorphism reflection (IR)

$$\frac{\Gamma \vdash e : A \cong B}{\Gamma \vdash A \equiv B}$$

Isomorphism reflection, so called because it reminds one of equality reflection (of which we shall have something to say shortly), states that isomorphic structures are judgmentally equal.

Because judgmental equality is a judgement form, the principle can only be stated as an inference rule (and not internally to type theory, like univalence).

Note that it does not matter whether we read $\cong$ as isomorphism or equivalence, because they are inter-derivable, and the conclusion forgets e.

Admittedly, it is an odd rule to consider. Is it even consistent? If we can equate, say, $A \times B$ and $B \times A$, it surely sounds like we are asking for trouble. We shall come back to this point, but let us first think about the consequences of isomorphism reflection.

Theorem: *Isomorphism reflection entails UIP.*

Proof. The type

$$\Pi\ (a, b : A)\ .\ \Pi\ (p : \mathrm{Id}_A(a,b))\ .\ \mathrm{Id}_{\mathrm{Id}_A(a,b)}(p, \mathrm{refl}_A(a))$$

is well-formed as $\mathrm{Id}_A(a,b) \cong \mathrm{Id}_A(a,a)$ using p, hence $\mathrm{Id}_A(a,b) \equiv \mathrm{Id}_A(a,a)$ by IR. It is inhabited by path induction:

$$(\lambda(a{:}A)\ .\ \mathrm{refl}_{\mathrm{Id}_A(a,a)}(\mathrm{refl}_A(a))) : \Pi\ (a : A)\ .\ \mathrm{Id}_{\mathrm{Id}_A(a,a)}(\mathrm{refl}_A(a), \mathrm{refl}_A(a)),$$

Now given p, q : $\mathrm{Id}_A(a,b)$ they are both equal to $\mathrm{refl}_A(a)$, hence equal.

Isomorphism reflection entails Uniqueness of identity proofs.

So the first lesson is that we’re talking essentially about a **set-level** principle.

Theorem (Rijke): *If IR holds then equality reflection and judgmental unit extensionality are inter-derivable:*

$$\frac{\Gamma \vdash p : \mathrm{Id}_A(a, b)}{\Gamma \vdash a \equiv_A b} \qquad \frac{\Gamma \vdash u : \mathrm{unit} \qquad \Gamma \vdash v : \mathrm{unit}}{\Gamma \vdash u \equiv_{\mathrm{unit}} v}$$

Proof (unit extensionality ⇒ equality reflection).

$\Sigma(x{:}A)\,.\,\mathrm{Id}_A(a, x)$ is contractible, therefore equivalent to unit, so equal to unit by IR. Hence $(a, \mathrm{refl}_A(a)) \equiv (b, p)$ by unit extensionality, from which we get

$$a \equiv \pi_1(a, \mathrm{refl}_A(a)) \equiv \pi_1(b, p) \equiv b.$$

The next observation is by Egbert Rijke (he literally converted a cup of morning coffee into this theorem – I was there).

In the presence of isomorphism reflection, equality reflection (the rule on the left) is inter-derivable with having a strict unit type (the rule on the right). The interesting direction goes from right to left. If unit is strict then so are all contractible types, because they're judgmentally equal to it. Now apply the observation to the contractible type Σ(x:A) . IdA(a, x) and project back to A.

- $(42, \text{false}) : \text{Nat} \times \text{Bool}$
- $\text{Nat} \times \text{Bool} \equiv \text{Bool} \times \text{Nat}$ (by IR)
- $(42, \text{false}) : \text{Bool} \times \text{Nat}$
- $\pi_1(42, \text{false}) : \text{Bool}$
- $42 : \text{Bool}$?!

Isomorphism reflection seemingly entails strange things, such as the one shown here.

- (42, false) : Nat × Bool
- Nat × Bool ≡ Bool × Nat   (by IR)
- (42, false) : Bool × Nat
- π_1(42, false) : Bool
- 42 : Bool ?!

$\pi_1(42, \text{false}) \not\equiv_{\text{Bool}} 42$

The mystery disappears when we observe that the last step is invalid. The pair (42, false) is formed as an element of type Nat × Bool, but π_1 is the projection from Bool × Nat.

With fully annotated the projections and pairs, we would see that the β-reduction step is not warranted.

Models

So far we do not know whether isomorphism reflection is even consistent. As we just saw, we have to be a bit careful with notation when using it.

The set-theoretic model

- Category of contexts and substitutions: **Set**
- Types are set families, $\mathrm{Fam}_\Gamma := \Gamma \to \mathbf{Set}$
- Terms are choice functions, $\mathrm{Term}_{\Gamma, A} := \Pi_\Gamma A$
- Reindexing is precomposition
- Context extension is coproduct: $\Gamma \,.\, A := \Sigma_\Gamma A$

Consider the set-theoretic model of type theory, presented as a category with families.

It invalidates isomorphism reflection because isomorphic sets need not be equal.

The cardinal model

- Category of contexts and substitutions: **Set**
- Types are set families, $\mathrm{Fam}_\Gamma := \Gamma \to$ **Card**
- Terms are choice functions, $\mathrm{Term}_{\Gamma, A} := \Pi_\Gamma A$
- Reindexing is precomposition
- Context extension is coproduct: $\Gamma . A := \Sigma_\Gamma A$

What if we replace families of sets with families of cardinal numbers? It works.

The cardinal model

- **Card** – the class of von Neumann cardinals
- Every set A is isomorphic to a unique cardinal $|A| \in$ **Card**.
- Using global choice, pick bijections $\xi_A : A \to |A|$.
- Type-theoretic constructions are rectified using $|\square|$ and ξ:
  - $\Pi^{Card}{}_{(x\,:\,A)}\, B(x) := |\Pi^{Set}{}_{(x\,:\,A)}\, B(x)|$
  - $(\lambda^{Card}\, t)\ \gamma = \xi\ (\lambda^{Set}\ (x \mapsto t\ (\gamma, x))$

You can read the detailed verification in Théo’s thesis.

A strange equation

- Cardinal model validates
  Nat → Nat ≡ Nat → Bool
- (λ n . n) : Nat → Nat
- (λ n . n) : Nat → Bool
- (λ n . n) 42 : Bool

Once again, we should not fall prey to working with terms that are not fully annotated.

This example is valid, but one has to keep in mind that the last line is not a beta β-redex because the λ-abstraction is at Nat → Nat and the application at Nat → Bool. It is *not* sufficient that these two types are equal, for the redex we need their domains and codomains to be equal, but the codomains differ.

Skeletal assemblies

- **Asm**($\mathbb{A}$) – assemblies on a pca $\mathbb{A}$
- **Skel**($\mathbb{A}$) – a skeleton of **Asm**($\mathbb{A}$)
- For each $A \in$ **Asm**($\mathbb{A}$), choose iso $\xi_A : A \to |A|$ to (unique) skeletal assembly $|A|$.
- Mimic the cardinal model.

We can use the passage to a skeletal category in other cases as well.

Consistency of IR

- Isomorphism reflection is consistent with (separately):
  - Markov principle & dependent choice
  - “All functions are continuous” (function realizability)
  - “All functions are computable” and
    Nat → Bool ≡ Nat → Nat (number realizability)

Consequently it follows that IR is consistent with various other principles.

Not that in the effective topos Nat → Bool and Nat → Nat are again isomorphic (even homeomorphic as metric spaces), so we again get the strange equation.

So what?

- Isomorphism reflection is a set-level concept.
- **Conjecture:** If $\mathcal{T}$+UIP is consistent then $\mathcal{T}$+IR is consistent.
- Can we implement limited forms of IR?
  - $\mathrm{Unary}\mathbb{N} \equiv \mathrm{Binary}\mathbb{N}$
  - $A \times (B \times C) \equiv (A \times B) \times C$?

What have we learned?

First of all, that isomorphism reflection is a set-level concept. And of course that it is strange.

Second, here is a conjecture that I do not know the answer to. If true, it says that type theory is quite permissive about judgmental equality.

Third and last, while it is fun to consider such a strange principle, we can also ask whether it is useful for anything?