The pullback lemma in gory detail

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May 30, 2012

A poor soul on the internet asked for the proof of the pullback lemma, which in every book on category theory is left as an exercise. I took pity on and wrote down the proof in gory detail.

Suppose in the following diagram the two squares are pullbacks:

\[
\begin{array}{ccc}
A & \xrightarrow{f} & B & \xrightarrow{g} & C \\
\downarrow{p} & & \downarrow{q} & & \downarrow{r} \\
X & \xrightarrow{u} & Y & \xrightarrow{v} & Z
\end{array}
\]

We would like to show that the outer rectangle is a pullback. For this purpose, consider the diagram

\[
\begin{array}{ccc}
Q & \xrightarrow{i} & B & \xrightarrow{r} & C \\
\downarrow{j} & & \downarrow{q} & & \downarrow{r} \\
X & \xrightarrow{u} & Y & \xrightarrow{v} & Z
\end{array}
\]

in which \(r \circ i = v \circ u \circ j\). Because \(v \circ (u \circ j) = r \circ i\), by the universal property of the right-hand square there exists a unique \(m : Q \to B\) such that \(g \circ m = i\) and \(q \circ m = u \circ j\):

\[
\begin{array}{ccc}
Q & \xrightarrow{m} & B & \xrightarrow{g} & C \\
\downarrow{j} & & \downarrow{q} & & \downarrow{r} \\
X & \xrightarrow{u} & Y & \xrightarrow{v} & Z
\end{array}
\]

Now by the universal property of the left-hand pullback there exists a unique
$n : Q \to A$ such that $f \circ n = m$ and $p \circ n = j$:

We claim that $n$ is the morphism we are looking for. Indeed, we already have $p \circ n = j$, and also

$$g \circ f \circ n = g \circ m = i.$$ 

It remains to show uniqueness of $n$. Suppose $n' : Q \to A$ satisfies $p \circ n' = j$ and $g \circ f \circ n' = i$. Let $m' = f \circ n'$, and observe that $g \circ m' = g \circ f \circ n' = i$ and $q \circ m' = q \circ f \circ n' = u \circ p \circ n' = u \circ j$, therefore by the uniqueness property of the right-hand square we get $m = m' = n' \circ f$. Now by the uniqueness property for the left-hand square we get $n = n'$, as desired.