Stone Duality for Skew Boolean Algebras with Intersections

Andrej Bauer
Faculty of Mathematics and Physics
University of Ljubljana
Andrej.Bauer@andrej.com

Karin Cvetko-Vah
Faculty of Mathematics and Physics
University of Ljubljana
Karin.Cvetko@fmf.uni-lj.si

May 30, 2011

Abstract

We extend Stone duality between generalized Boolean algebras and Boolean spaces, which are the zero-dimensional locally-compact Hausdorff spaces, to a non-commutative setting. We first show that the category of right-handed skew Boolean algebras with intersections is dual to the category of surjective étale maps between Boolean spaces. We then extend the duality to skew Boolean algebras with intersections, and consider several variations in which the morphisms are restricted. Finally, we use the duality to construct a right-handed skew Boolean algebra without a lattice section.

1 Introduction

The fundamental example of the kind of duality we are interested in was established by Marshall Stone [9, 10]: every Boolean algebra corresponds to a zero-dimensional compact Hausdorff space, or a *Stone space* for short, as well as to a *Boolean ring*, which is a commutative ring of idempotents with a unit. In modern language the duality is stated as equivalence of categories of Boolean algebras, Boolean rings, and Stone spaces, where the later equivalence is contravariant. The duality has many generalizations, see [3]. Already in Stone's second paper [10, Theorem 8] we find an extension of duality to *Boolean spaces*, which are the zero-dimensional *locally* compact Hausdorff spaces. They correspond to commutative rings of idempotents, possibly without a unit, or equivalently to *generalized Boolean algebras*, which are like Boolean algebras without a top element.

Our contribution to the topic is a study of the *non-commutative* case. Among several variations of non-commutative Boolean algebras we are able to provide duality for *skew Boolean algebras with intersections* (which we call *skew algebras*) because they have a well-behaved theory of ideals.

The paper is organized as follows. In Section 2 we recall the necessary background material about skew Boolean algebras, Boolean spaces, and étale maps. In Section 3 we spell out the well-known Stone duality for commtutative algebras. In Section 4 we establish the duality between right-handed skew algebras and skew Boolean spaces, which we then extend to the duality between skew algebras and rectangular skew Boolean spaces. In Section 5 we further analyze the situation and consider several variations of the duality in which morphisms are restricted. In Section 6 we use the duality to construct a right-handed skew algebra without a lattice section. This answers negatively a hitherto open question of existence of such algebras.

Acknowledgment. We thank Jeff Egger, Mai Gehrke, Ganna Kudryavtseva, Jonathan Leech, and Alex Simpson for discussing the topic with us and offering valuable advice.

2 Preliminary definitions

In the first part of the section we review basic concepts and notation regarding skew Boolean algebras. In the second part we recall some basic facts about Boolean spaces and étale maps.

2.1 Skew Boolean algebras

A *skew lattice* is an algebra $(A, \wedge, \vee)$ with idempotent and associative binary operations *meet* $\wedge$ and *join* $\vee$ satisfying the absorption identities $x \wedge (x \vee y) = x = (y \vee x) \wedge x$ and $x \vee (x \wedge y) = x = (y \wedge x) \vee x$. If one of the operations is commutative then so is the other, in which case A is a lattice, see [5].

A skew lattice has two order structures. The *natural partial order* $x \leq y$ is defined by $x \wedge y = y \wedge x = x$, or equivalently $x \vee y = y \vee x = y$. The *natural preorder* $x \preceq y$ is defined by $x \wedge y \wedge x = x$, or equivalently $y \vee x \vee y = y$. The poset refelection of the natural preorder $\preceq$ is known as *Green's relation* $\mathcal{D}$. By Leech's First Decomposition Theorem [5], $\mathcal{D}$ is the finest congruence for which $A/\mathcal{D}$ is a lattice. In other words, the functor $A \mapsto A/\mathcal{D}$ is a reflection of skew lattices into ordinary lattices. We denote the $\mathcal{D}$-equivalence class of a by $\mathcal{D}_a$.

The reflection can be analysed further into its left- and right-handed parts. A skew lattice A is *right-handed* if it satisfies the identity $x \wedge y \wedge x = y \wedge x$, or equivalently $x \vee y \vee x = x \vee y$. We define *left-handed* lattices analogously. Quotients by Green's congruence relations $\mathcal{R}$ and $\mathcal{L}$, which are defined by

$$\begin{aligned} x \,\mathcal{R}\, y &\iff x \wedge y = y \text{ and } y \wedge x = x, \\ x \,\mathcal{L}\, y &\iff x \wedge y = x \text{ and } y \wedge x = y, \end{aligned}$$

provide reflections of a skew lattice A into left-handed and right-handed skew lattices, respectively. By Leech's Second Decomposition Theorem [5] the square of canonical quotient maps

$$\begin{array}{ccc} A & \longrightarrow & A/\mathcal{L} \\ \downarrow & & \downarrow \\ A/\mathcal{R} & \longrightarrow & A/\mathcal{D} \end{array}$$

is a pullback in the category of skew algebras.

A *rectangular band* $(A, \wedge)$ is an algebra with a binary operation $\wedge$ which is idempotent, associative, and it satisfies the rectangle identity $x \wedge y \wedge z = x \wedge z$. The name comes from the fact that every rectangular band is isomorphic to a cartesian product $X \times Y$ with the operation $(x_1, y_1) \wedge (x_2, y_2) = (x_1, y_2)$. If A is non-empty the sets X and Y are unique up to bijection and can be taken to be $A/\mathcal{L}$ and $A/\mathcal{R}$, respectively. Each rectangular band is a skew lattice for $\wedge$ and the associated operation $\vee$ defined as $x \vee y = y \wedge x$. It turns out that the $\mathcal{D}$-classes of a skew lattice form rectangular bands, with the operation induced by the skew lattice.

A *skew Boolean algebra* $(A, 0, \wedge, \vee, \setminus)$ is a skew lattice which is meet-distributive, i.e., it satisfies the identities

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z) \quad \text{and} \quad (y \vee z) \wedge x = (y \wedge x) \vee (z \wedge x),$$

has a *zero* 0, which is neutral for $\vee$, and a *relative complement* $\setminus$ satisfying

$$(x \setminus y) \wedge (x \wedge y \wedge x) = 0 \quad \text{and} \quad (x \setminus y) \vee (x \wedge y \wedge x) = x.$$

It follows from these requirement that the principal subalgebras $x \wedge A \wedge x$ of a skew Boolean algebra are Boolean algebras. For example, the two identities for the relative complement say that $\setminus$ restricted to a principal subalgebra acts as the complement operation. See Leech [6] for further details on skew Boolean alebras. We remark that a skew Boolean algebra with a top element 1 is degenerate in the sense that it is already a Boolean algebra. Also note that a skew Boolean algebra whose meet and join are commutative is the same thing as a generalized Boolean algebra.

Often it is the case that any two elements x and y of a skew Boolean algebra A have the greatest lower bound $x \cap y$ with respect to the natural partial order $\leq$. When this is the case, we call $\cap$ *intersection* and speak of a *skew Boolean intersection algebra*. Many examples of skew lattices occurring in nature posses intersections. The significance of skew Boolean intersection algebras is witnessed by the fact that they form a discriminator variety [1], and are therefore both congruence permutable and congruence distributive. Moreover, results by Bignall and Leech [1] imply that every algebra A in a pointed discriminator variety is term equivalent to a right-handed skew Boolean intersection algebra whose congruences coincide with those of A. In contrast, it was observed already by Cornish [2] that the congruence lattices of skew Boolean algebras in general satisfy no particular lattice identity.

Henceforth we shall consider exclusively skew Boolean intersection algebras, so we simply refer to them as *skew algebras*. A homomorphism of skew algebras preserves all the operations, namely 0, $\wedge$, $\vee$, $\setminus$, and $\cap$.

Recall that an *ideal*, which we sometimes call $\preceq$*-ideal*, in a skew algebra A is a subset $I \subseteq A$ which is lower with respect to $\preceq$ and is closed under finite joins, so in particular $0 \in I$. An ideal P is *prime* if it is non-trivial and $a \wedge b \in P$ implies $a \in P$ or $b \in P$. It can be shown easily that the prime ideals in A coincide with non-zero maps $A \to 2$ into the two-elments lattice $2 = \{0, 1\}$ which preserve 0, $\wedge$, and $\vee$ (but not necessarily $\cap$). Because 2 is commutative, such maps are in bijective correspondence with non-zero maps $A/\mathcal{D} \to 2$. In other words, the assignment

$$P \mapsto P/\mathcal{D} = \{\mathcal{D}_a \mid a \in P\}$$

is a bijection from prime ideals in A to prime ideals in $A/\mathcal{D}$.

We write $f : X \rightharpoonup Y$ to indicate that f is a partial map from X to Y, defined on its *domain* $\mathrm{dom}(f) \subseteq X$. The *restriction* $f|_D$ of $f : X \rightharpoonup Y$ to $D \subseteq X$ is the map f with the domain restricted to $\mathrm{dom}(f) \cap D$. We denote the set of all partial maps from X to Y by $\mathcal{P}(X, Y)$. Leech's construction [6] shows how $\mathcal{P}(X, Y)$ can be endowed with a right-handed skew algbera structure by setting

$$\begin{aligned} 0 &= \emptyset, \\ f \wedge g &= g|_{\mathrm{dom}(f) \cap \mathrm{dom}(g)}, \\ f \vee g &= f \cup g|_{\mathrm{dom}(f) - \mathrm{dom}(g)}, \\ f \setminus g &= f|_{\mathrm{dom}(f) - \mathrm{dom}(g)}, \\ f \cap g &= f \cap g, \end{aligned}$$

where $-$ is set-theoretic difference, and the set-theoretic operations on the right-hand sides act on f and g viewed as functional relations. We generalize this construction to algebras which are not necessarily right-handed, because we will need one in Section 4.4.

Given a subset $D \subseteq X$, let $\mathcal{P}(X, Y)_D = \{f : X \rightharpoonup Y \mid \mathrm{dom}(f) = D\}$ be the set of those partial maps $X \rightharpoonup Y$ whose domain is D. Suppose we are given for each $D \subseteq X$ a binary operation $\curlywedge_D$ on $\mathcal{P}(X, Y)_D$. We say that the family $\{\curlywedge_D \mid D \subseteq X\}$ is *coherent* when it commutes with restrictions: for all $E \subseteq D \subseteq X$ and $f, g \in \mathcal{P}(X, Y)_D$ it holds

$$(f \curlywedge_D g)|_E = f|_E \curlywedge_E g|_E. \tag{1}$$

We usually omit the subscript from $\curlywedge_D$ and write just $\curlywedge$.

Theorem 2.1 *Let $(\curlywedge_D)_{D \subseteq X}$ be a coherent family of rectangular bands. Then $\mathcal{P}(X, Y)$ is a skew algebra for the following operations, defined for $f, g \in \mathcal{P}(X, Y)$ with $\mathrm{dom}(f) = F$ and $\mathrm{dom}(g) = G$:*

$$\begin{aligned} 0 &= \emptyset, \\ f \wedge g &= f|_{F \cap G} \curlywedge g|_{F \cap G}, \\ f \vee g &= (f|_{F-G}) \cup (g|_{G-F}) \cup (g \wedge f), \\ f \setminus g &= f|_{F-G}, \\ f \cap g &= f \cap g. \end{aligned}$$

Proof. Let $f, g, h \in \mathcal{P}(X, Y)$ be partial maps with domains $\mathrm{dom}(f) = F$, $\mathrm{dom}(g) = G$, and $\mathrm{dom}(h) = H$, respectively. Observe that the operations defined in the statement of the theorem all commute with restrictions, for example

$$\begin{aligned} (f \vee g)|_D = f|_{(F-G) \cap D} \cup g|_{(G-F) \cap D} \cup (f|_{D \cap F \cap G} \curlywedge g|_{D \cap F \cap G}) = \\ f|_{(D \cap F) - (D \cap G)} \cup g|_{(D \cap G) - (D \cap F)} \cup (f|_{D \cap F \cap G} \curlywedge g|_{D \cap F \cap G}) = f|_D \vee g|_D. \end{aligned}$$

Thus a good strategy for checking an identity is to do it "by parts" as follows. To check $u = v$ it suffices to check $u|_X = v|_X$ and $u|_Y = v|_Y$ separately, provided that $\mathrm{dom}(u) = \mathrm{dom}(v) = X \cup Y$. Of course, this only works if X and Y are suitably chosen so that the restrictions $u|_X$, $u|_Y$, $u|_Y$, and $v|_Y$ simplify when we push the restrictions by X and Y inwards.

The following properties are easily verified: 0 is neutral for $\vee$, idempotency of $\wedge$ and $\vee$, associativity of $\wedge$. That $\cap$ computes greatest lower bounds holds because the natural partial order on $\mathcal{P}(X, Y)$ is subset inclusion $\subseteq$ of functions viewed as functional relations. It remains to check associativity of $\vee$, meet distributivity, and the properties of $\setminus$.

Associativity of $\vee$ is checked by parts. The domain of $f \vee (g \vee h)$ and $(f \vee g) \vee h$ is $F \cup G \cup H$, which is covered by the parts $(F \cup G) - H$, $(F \cup H) - G$, $(G \cup H) - F$, and $F \cap G \cap H$. On the first part we get

$$((f \vee g) \vee h)|_{(F \cup G) - H} = (f|_{F-H} \vee g|_{G-H}) \vee h|_\emptyset = f|_{F-H} \vee g|_{G-H}.$$

and

$$(f \vee (g \vee h))|_{(F \cup G) - H} = f|_{F-H} \vee (g|_{G-H} \vee h|_\emptyset) = f|_{F-H} \vee g|_{G-H}.$$

On $(F \cup H) - G$ and $(G \cup H) - F$ the calculation is similar, while on $F \cap G \cap H$ both meets and joins turn into $\curlywedge$ and the identity follows as well.

Next we check that meet distributivity holds. The domain of $f \wedge (g \vee h)$ and $(f \wedge g) \vee (f \wedge h)$ is $F \cap (G \cup H)$. It is covered by the parts $(F \cap G) - H$, $(F \cap H) - G$, and $F \cap G \cap H$. On the first part we get

$$(f \wedge (g \vee h))|_{(F \cap G) - H} = f|_{(F \cap G) - H} \wedge (g|_{(F \cap G) - H} \vee h|_\emptyset) = \\ f|_{(F \cap G) - H} \wedge g|_{(F \cap G) - H}$$

and

$$(f \wedge g) \vee (f \wedge h))|_{(F \cap G) - H} = \\ (f|_{(F \cap G) - H} \wedge g|_{(F \cap G) - H}) \vee (f|_{(F \cap G) - H} \wedge h|_\emptyset) = \\ (f|_{(F \cap G) - H} \wedge g|_{(F \cap G) - H}) \vee 0 = f|_{(F \cap G) - H} \wedge g|_{(F \cap G) - H}.$$

The calculation on $(F \cap H) - G$ is similar, and on $F \cap G \cap H$ it again trivializes because all operations become $\curlywedge$. For the other half of meet distributivity, namely $(f \vee g) \wedge h = (f \wedge h) \vee (g \wedge h)$ we use the parts $(F \cap H) - G$, $(G \cap H) - F$, and $F \cap G \cap H$.

Finaly, $\setminus$ satisfies the axioms of relative complementation because

$$(f \setminus g) \wedge (f \wedge g \wedge f) = f|_{F-G} \wedge f|_{F \cap G} = \emptyset = 0$$

and

$$(f \setminus g) \vee (f \wedge g \wedge f) = f|_{F-G} \vee f|_{F \cap G} = f|_{F-G} \cup f|_{F \cap G} \cup \emptyset = f.$$

□

2.2 Boolean spaces and étale maps

We start by recaling several standard topological notions. A space is *zero-dimensional* if its clopens (sets which are both open and closed) form a topological base. A *Stone space* is a compact zero-dimensional Hausdorff space, while a *Boolean space* is a locally compact zero-dimensional Hausdorff space. We call a set which is compact and open a *copen*. In a Boolean space the copens form toplogical base.

A partial map $f : X \rightharpoonup Y$ is said to be continuous when it is continuous as a map defined on the subset $\mathrm{dom}(f) \subseteq X$ with the induced topology. Unless noted otherwise, the domain of definition $\mathrm{dom}(f)$ is always going to be an open subset of X. A continuous map is *proper* if its inverse image map takes compact subsets to compact subsets, while a partial continuous map with an open domain of definition is proper when the inverse image $f^{-1}(K)$ of a compact subset $K \subseteq Y$ is compact in $\mathrm{dom}(f)$, or equivalently in X.

An *étale map* $p : E \to B$, also known as *local homeomorphism*, is a continuous map for which E has an open cover such that for each U in the cover the restriction $p|_U : U \to p(U)$ is a homeomorphism onto the image, and $p(U)$ is open in B. We call E the *total space* and B the *base* of the étale map p. The *fiber* above $x \in B$ is the subspace $E_x = \{y \in E \mid p(y) = x\}$. A *section* of p is a continuous map $s : U \to E$ defined on a subset $U \subseteq B$, usually open, such that $p \circ s = \mathrm{id}_U$.

Étale maps with a common base B form a category, even a topos, in which morphisms are commutative triangles

$$\begin{array}{ccc} E & \xrightarrow{\quad f \quad} & E' \\ & {\scriptstyle p}\searrow \qquad \swarrow{\scriptstyle p'} & \\ & B & \end{array}$$

where f is a continuous map. It follows that f is an étale map, see for example [7, II.6]. One consequence of this is that a section $s : U \to E$ of an étale map $p : E \to B$ defined on an open subset $U \subseteq B$ is itself an étale map (because the inclusion $U \hookrightarrow E$ is an étale map). In particular, the image $s(U)$ is open in E and (images of) open sections form a base for E.

A *copen section* of $p : E \to B$ is a section $s : U \to E$ defined on a copen subset $U \subseteq B$. Its image $s(U)$ is not only open but also compact in E, and the restriction $p|_{s(U)}$ is a homeomorphism from $s(U)$ onto U. Conversely, if $S \subseteq E$ is copen and $p|_S : S \to p(S)$ is a homeomorphism onto $p(S)$ then $(p|_S)^{-1} : p(S) \to S$ is a copen section. This is so because étale maps are open. We therefore have two views of copen sections: as sections defined on copen subsets, and as those copen subsets of the total space which cover each point in the base at most once.

In Section 4.4 we will need to know how to compute equalizers and coequalizers in the category of étale maps over a given base.

Proposition 2.2 *Let f and g be morphisms of étale maps,*

$$\begin{array}{ccc} E & \overset{f}{\underset{g}{\rightrightarrows}} & E' \\ & {\scriptstyle p}\searrow \qquad \swarrow{\scriptstyle p'} & \\ & B & \end{array}$$

The equalizer and coequalizer of f and g in the category of étale maps with base B are computed as in the category of topological spaces. Moreover, the quotient map from E' to the coequalizer is open.

Proof. In the category of topological spaces the equalizer of f and g is the subspace

$$I = \{x \in E \mid f(x) = g(x)\}$$

with the subspace inclusion $i : I \to E$. For this to be an equalizer in the category of étale maps, $p \circ i : I \to B$ must be étale, which is the case because I is an open subspace of E. To see this, consider any $x \in I$. Because f and g are étale maps there is an open section $U \subseteq E$ containing x such that $f|_U : U \to f(U)$ and $g|_U : U \to g(U)$ are homeomorphisms onto open subsets of E'. Thus the intersection $f(U) \cap g(U)$ is open, from which it follows that $U \cap f^{-1}(g(U)) \cap g^{-1}(f(U))$ is an open neighborhood of x contained in I.

The coequalizer of f and g is computed in topological spaces as the quotient space $Q = E'/R$ where $R \subseteq E' \times E'$ is the least equivalence relation generated by the relation $S = \{(f(x), g(x)) \in E' \times E' \mid x \in E\}$. Because S is an open subset of the fibered product $E' \times_B E'$ so is R, from which it follows that the canonical quotient map $q : E' \to Q$ is open. Furthermore, because $p' \circ f = p = p' \circ g$ the map p' factors through q as $p' = r \circ q$. To complete the proof, we need to show that r is étale, but this is easy because we already know that q is open. □

3 Duality for commutative algebras

Before embarking on duality for skew algebras we review the familiar commutative case. The *spectrum* $\mathsf{St}(A)$ of a Boolean algebra A is the Stone space whose points are the prime ideals of A. The elements $a \in A$ correspond to the basic clopen sets $\mathcal{N}_a = \{P \in \mathsf{St}(A) \mid a \notin P\}$. A homomorphism $f : A \to A'$ between Boolean algebras induces a continuous map $f_\flat : \mathsf{St}(A') \to \mathsf{St}(A)$ that maps a prime ideal P to its preimage $f_\flat(P) = f^{-1}(P)$. In the other direction the duality maps a Stone space to the Boolean algebra of its clopen subsets, with the expected operations of intersection and union.

A short path to Stone duality for generalized Boolean algebras goes through the observation that the category GBA of generalized Boolean algebras is equivalent to the slice category $\mathsf{BA}/2$ of Boolean algebras over the initial algebra 2. By Stone duality for Boolean algebras GBA is then dual to *pointed* Stone spaces and continuous maps which preserve the chosen point. For our purposes it is more convenient to take yet another equivalent category, namely Stone spaces *without* one point, which are precisely the Boolean spaces, and suitable partial maps between them.

Let us describe the duality explicitly. Starting from a generalized Boolean algebra A, we construct its *spectrum* $\mathsf{St}(A)$ as the Boolean space of prime ideals. An element $a \in A$ corresponds to the basic copen set $\mathcal{N}_a = \{P \in \mathsf{St}(A) \mid a \notin P\}$. A homomorphism $f : A \to A'$ between generalized Boolean algebras induces a partial map $f_\flat : \mathsf{St}(A') \rightharpoonup \mathsf{St}(A)$ that maps a prime ideal P to its preimage $f_\flat(P)$, provided the preimage is not all of A. The domain of definition of $f_\flat$ is open, for if $a \notin f_\flat(P)$ then $f_\flat$ is defined on $\mathcal{N}_{f(a)}$. The fact that the inverse image $f_\flat^{-1}(\mathcal{N}_a)$ of a basic copen is the basic copen $\mathcal{N}_{f(a)}$ implies that $f_\flat$ is both continuous and proper. Indeed, f is proper because the inverse image $f^{-1}(K)$ of a compact subset $K \subseteq \mathsf{St}(A)$ is a closed subset of the compact set $f^{-1}(\mathcal{N}_{a_1}) \cup \cdots \cup f^{-1}(\mathcal{N}_{a_n})$ where $\mathcal{N}_{a_1}, \ldots, \mathcal{N}_{a_n}$ is some finite cover of K by basic copen sets. In summary, the category of generalized Boolean algebras is equivalent to the category of Boolean spaces and proper continuous partial maps with open domains of definition.

4 Duality for skew algebras

Any attempt at extension of Stone duality naturally leads to consideration of prime ideals. Since the one-point space 1 corresponds to the initial Boolean algebra 2, a point $1 \to X$ on the topological side corresponds to homomorphisms $A \to 2$ on the algebraic side. However, since such homomorphisms factor through the commutative reflection $A \to A/\mathcal{D}$, they give us insufficient information about the non-commutative structure of A. We should therefore expect that on the topological side we have to look for structures that can be rich even though they have few *global* points, while on the algebraic side we cannot afford to use ideals exclusively, but must also consider congruence relations.

In the case of a commutative algebra the congruence relation generated by a prime ideal has just two equivalence classes. In contrast, the least congruence relation θ_P whose zero-class contains the prime ideal $P \subseteq A$ in a skew algebra A generally has many equivalence classes, which ought to be accounted for on the topological side of duality. The following result of Bignall and Leech [1] characterizes the congruence relations generated by ideals in a skew algebra.

Lemma 4.1 (Bignall & Leech) *For an ideal $I \subseteq A$ in a skew algebra A let θ_I be the least congruence relation on A whose zero-class contains I. Then for all $x, y \in A$, $x \,\theta_I\, y$ if, and only if, $(x \setminus (x \cap y)) \vee (y \setminus (x \cap y)) \in I$.*

In fact, the zero-class of θ_I equals I. Consequently, the ideals of a skew algebra are in bijective correspondence with congruence relations. A consequence of Lemma 4.1 is that $x \,\theta_{f^{-1}(I)}\, y$ is equivalent to $f(x) \,\theta_I\, f(y)$, for any skew algebra homomorphism $f : A \to A'$ and any ideal $I \subseteq A'$. From this we obtain the following properties of prime ideals.

Lemma 4.2 *Let P be a prime ideal in a skew algebra A and let $x, y \in A$. Then:*

1. *$x \in P$ or $y \setminus x \in P$.*
2. *If $x \leq y$ and $x \notin P$ then $x \,\theta_P\, y$.*
3. *If $x \leq y$ then $x \in P$ is equivalent to $y \,\theta_P\, y \setminus x$.*

Proof.

1. We have $x \wedge (y \setminus x) = 0 \in P$. Because P is a prime ideal it follows that $x \in P$ or $y \setminus x \in P$.
2. We need to show that $(x \setminus (x \cap y)) \vee (y \setminus (x \cap y)) \in P$. Since $x \leq y$ it follows that $x \cap y = x$ and thus we only need $y \setminus x \in P$, which follows from the first statement of the lemma.
3. The element $x' = y \setminus x$ is the complement of x in the Boolean algebra $y \wedge A \wedge y$. Hence $x' \leq y$ and $y \setminus x' = x$. Thus Lemma 4.1 implies that $y \,\theta_P\, x'$ is equivalent to $(y \setminus x') \vee (x' \setminus x') = x \in P$.

□

4.1 From algebras to spaces

Following our own advice that the topological side of duality should account for the equivalence classes of congruences, we define the *skew spectrum* $\mathsf{Sk}(A)$ of a skew algebra A to be the space whose points are pairs (P, e), where P is a prime ideal in A and e is a non-zero equivalence class of θ_P. Since every non-zero equivalence class equals $[t]_{\theta_P}$ for some $t \notin P$, a general element of the skew spectrum may be written as $(P, [t]_{\theta_P})$. We write just $[t]_P$. Thus, as a set the skew spectrum is

$$\mathsf{Sk}(A) = \{[t]_P \mid P \text{ prime ideal in } A \text{ and } t \notin P\},$$

where $[t]_P = [u]_Q$ when $P = Q$ and $t\,\theta_P\,u$. The topology of $\mathsf{Sk}(A)$ is the one whose basic open sets are of the form, for $a \in A$,

$$\mathcal{M}_a = \{[a]_P \mid a \notin P\}.$$

These really form a basis because they are closed under intersections.

Lemma 4.3 *Let $a, b \in A$. Then $\mathcal{M}_a \cap \mathcal{M}_b = \mathcal{M}_{a\cap b}$.*

Proof. For a prime ideal P and $t \in A$ the statement $[t]_P \in \mathcal{M}_a \cap \mathcal{M}_b$ amounts to

$$(a \notin P) \wedge (b \notin P) \wedge (t\,\theta_P\,a) \wedge (t\,\theta_P\,b), \tag{2}$$

while $[t]_P \in \mathcal{M}_{a\cap b}$ means

$$(a \cap b \notin P) \wedge (t\,\theta_P\,a \cap b). \tag{3}$$

If (2) holds then $a \cap b\,\theta_P\,a \cap a = a\,\theta_P\,t$ which proves (3).

To prove the converse, suppose (3) holds. Because $a \cap b \notin P$, $a \cap b \leq a$ and $a \cap b \leq b$, Lemma 4.2 implies $a\,\theta_P\,a \cap b\,\theta_P\,b$ which suffices for (2). □

The skew spectrum on its own contains too little information to act as the dual. For example, both $\mathsf{Sk}(2\times 2)$ and $\mathsf{Sk}(3)$ are the discrete space on two points. Recall that $\mathsf{3}$ is the right-handed skew algebra whose set of elements is $\{0, 1, 2\}$ and whose $\mathcal{D}$-classes are $\{0\}$ and $\{1, 2\}$. One part of the missing information is provided by the map $q_A : \mathsf{Sk}(A) \to \mathsf{St}(A)$ defined by

$$q_A([t]_P) = P/\mathcal{D},$$

where we used the shorthand $\mathsf{St}(A) = \mathsf{St}(A/\mathcal{D})$.

Proposition 4.4 *The map $q_A : \mathsf{Sk}(A) \to \mathsf{St}(A)$ is onto and étale.*

Proof. Because prime ideals are non-trivial the map q_A is onto. To show that q_A is continuous, we prove

$$q_A^{-1}(\mathcal{N}_a) = \bigcup_{b \preceq a} \mathcal{M}_b, \tag{4}$$

where we used the shorthand $\mathcal{N}_a = \mathcal{N}_{\mathcal{D}_a}$. For one inclusion, observe that $b \preceq a$ implies $\mathcal{D}_b \leq \mathcal{D}_a$ and hence $q_A(\mathcal{M}_b) = \mathcal{N}_b \subseteq \mathcal{N}_a$. For the other inclusion, suppose $q_A([t]_P) \in \mathcal{N}_a$. Then $t \wedge a \wedge t \notin P$ because $a \notin P$ and $t \notin P$, and by Lemma 4.2 we get $t \,\theta_P\, t \wedge a \wedge t$ from which $[t]_P \in \mathcal{M}_{t\wedge a\wedge t}$ follows.

Finally, q_A is étale because its restriction to a basic open set $\mathcal{M}_a$ is a (continuous) bijection onto the basic open set $\mathcal{N}_a$. □

Corollary 4.5 *The skew spectrum* $\mathsf{Sk}(A)$ *is a Boolean space.*

Proof. Local compactness and zero-dimensionality are lifted from $\mathsf{St}(A)$ to $\mathsf{Sk}(A)$ by the étale map q_A. To see that $\mathsf{Sk}(A)$ is Hausdorff, let $[s]_P$ and $[t]_Q$ in $\mathsf{Sk}(A)$ be two distinct points in $\mathsf{Sk}(A)$.

Consider the case $P = Q$. We claim that $\mathcal{M}_{s\setminus(s\cap t)}$ and $\mathcal{M}_{t\setminus(s\cap t)}$ are disjoint and are neighborhoods of s and t, respectively. Disjointness follows by Lemma 4.3 from the fact that $(t \setminus (s \cap t)) \cap (s \setminus (s \cap t)) = 0$. From $\neg(t \,\theta_P\, s)$ we conclude by Lemma 4.1 that $(s \setminus (s \cap t)) \vee (t \setminus (s \cap t)) \notin P$, thus $s \setminus (s \cap t) \notin P$ or $t \setminus (s \cap t) \notin P$. In either case $s \cap t \in P$ by (1) of Lemma 4.2, therefore $s \,\theta_P\, s \setminus (s \cap t)$ by (3) of Lemma 4.2, which proves $[s]_P \in \mathcal{M}_{s\setminus(s\cap t)}$, as claimed. We similarly show that $[t]_P \in \mathcal{M}_{t\setminus(s\cap t)}$.

Consider the case $P \neq Q$. Because $\mathsf{St}(A)$ is Hausdorff there exist disjoint basic open nehigborhoods $\mathcal{N}_a$ and $\mathcal{N}_b$ of $q_A(P)$ and $q_A(Q)$, respectively. Thus $q_A^{-1}(\mathcal{N}_a)$ and $q_A^{-1}(\mathcal{N}_b)$ are disjoint and are open neighborhoods of $[s]_P$ and $[t]_Q$, respectively. □

Because Boolean spaces are zero-dimensional, the étale map $q_A : \mathsf{Sk}(A) \to \mathsf{St}(A)$ has more sections that one would normally expect.

Proposition 4.6 *The étale map* $q_A : \mathsf{Sk}(A) \to \mathsf{St}(A)$ *has a section above every copen set, passing through a prescribed point above the copen set.*

Proof. More precisely, the proposition claims that given a copen set $U \subseteq \mathsf{St}(A)$ and a point $x \in \mathsf{Sk}(A)$ such that $q_A(x) \in U$, there is a copen section above U containing x. For the proof we only need surjectivity of q_A and zero-dimensionality of $\mathsf{St}(A)$. By surjectivity the copen set U can be decomposed into pairwise disjoint copen sets $U_1, \ldots, U_n$, each of which has a section s_i. The sections can be glued together into a section s above U. To make sure that s passes through a prescribed point x, just change it suitably on a small copen neighborhood of $q_A(x)$. □

Proposition 4.7 *The copen sections of* $q_A : \mathsf{Sk}(A) \to \mathsf{St}(A)$ *are precisely the sets* $\mathcal{M}_a$ *with* $a \in A$.

Proof. We already know that each $\mathcal{M}_a$ is a copen section because q_A maps it homeomorphically onto the copen set $\mathcal{N}_a$. Conversely, let V be a copen section in $\mathsf{Sk}(A)$. It is a finite union of basic copen sets $V = \mathcal{M}_{a_1} \cup \cdots \cup \mathcal{M}_{a_n}$. We show that $V = \mathcal{M}_{a_1\vee\cdots\vee a_n}$. If $n = 1$ there is nothing to prove.

Consider the case $n = 2$. The fact that $V = \mathcal{M}_{a_1} \cup \mathcal{M}_{a_2}$ is a section amounts to: for all prime ideals P in A, if $a_1 \notin P$ and $a_2 \notin P$ then $a \,\theta_P\, b$. Let us show that this implies $\mathcal{M}_{a_1} \subseteq \mathcal{M}_{a_1\vee a_2}$. If $a_1 \notin P$ then $a_1 \vee a_2 \notin P$. Either $a_2 \in P$ or $a_2 \notin P$. In the first case $a_2 \,\theta_P\, 0$ and so $a_1 \,\theta_P\, a_1 \vee 0 \,\theta_P\, a_1 \vee a_2$. In the second case,

$a_2\,\theta_P\,a_1$ and so $a_1\,\theta_P\,a_1 \vee a_1\,\theta_P\,a_1 \vee a_2$. We similarly show that $\mathcal{M}_{a_2} \subseteq \mathcal{M}_{a_1\vee a_2}$, from which we get $\mathcal{M}_{a_1} \cup \mathcal{M}_{a_2} \subseteq \mathcal{M}_{a_1\vee a_2}$.

To complete the case $n = 2$ we still have to prove $\mathcal{M}_{a_1\vee a_2} \subseteq \mathcal{M}_{a_1} \cup \mathcal{M}_{a_2}$. Suppose $a_1 \vee a_2 \notin P$. Then either $a_1 \notin P$ or $a_2 \notin P$. We consider the case $a_1 \notin P$, the other one is similar. Either $a_2 \in P$ or $a_2 \notin P$. In the first case $a_2\,\theta_P\,0$ so $a_1\,\theta_P\,a_1 \vee 0\,\theta_P\,a_1 \vee a_2$. In the second case the assumption gives us $a_1\,\theta_P\,a_2$ and so $a_1\,\theta_P\,a_1 \vee a_1\,\theta_P\,a_1 \vee a_2$.

The cases $n > 2$ follow by repeated use of the case $n = 2$. □

Later on we will need to know how the map $a \mapsto \mathcal{M}_a$ interacts with the operations of A. For this purpose we define the *saturation* operation $\sigma_A(U) = q_A^{-1}(q_A(U))$. Since q_A is open, the saturation of a copen is a clopen.

Proposition 4.8 *The assignment $a \mapsto \mathcal{M}_a$ is an order-isomorphism from a skew algebra A, ordered by the natural partial order, onto the copen sections of $\mathsf{Sk}(A)$, ordered by subset inclusion, satisfying the identities*

$$\mathcal{M}_{a\wedge b\wedge a} = \sigma_A(\mathcal{M}_b) \cap \mathcal{M}_a \quad \textit{and} \quad \mathcal{M}_{a\vee b\vee a} = \mathcal{M}_a \cup (\mathcal{M}_b \setminus \sigma_A(\mathcal{M}_a)).$$

Proof. If $\mathcal{M}_a = \mathcal{M}_b$ then $a\,\theta_P\,b$ for all prime ideals P. Thus for any prime ideal P

$$(a \setminus (a \cap b)) \vee (b \setminus (a \cap b)) \in P,$$

and so both $a \setminus (a \cap b)$ and $b \setminus (a \cap b)$ belong to P. Because the intersection of all prime ideals is $\{0\}$, it follows that

$$a \setminus (a \cap b) = 0 = b \setminus (a \cap b).$$

Therefore, $\mathcal{D}_a = \mathcal{D}_{a\cap b} = \mathcal{D}_b$, which is only possible if $a = b$. It follows that the assignment $a \mapsto \mathcal{M}_a$ is injective, while its surjectivity follows from Proposition 4.7.

That $a \mapsto \mathcal{M}_a$ is an order isomorphism follows from the following chain of equivalences, where we use Lemma 4.3 and injectivity of $a \mapsto \mathcal{M}_a$ in the second and third step, respectively:

$$\begin{aligned} a \leq b \iff a \cap b = a \iff \mathcal{M}_{a\cap b} = \mathcal{M}_a \\ \iff \mathcal{M}_a \cap \mathcal{M}_b = \mathcal{M}_a \iff \mathcal{M}_a \subseteq \mathcal{M}_b. \end{aligned}$$

It remains to prove the two identites. For the first one, let $[a\wedge b\wedge a]_P \in \mathcal{M}_{a\wedge b\wedge a}$. Then $a \wedge b \wedge a \notin P$ which implies both $b \notin P$ (and thus $[a \wedge b \wedge a]_P \in \sigma_A(\mathcal{M}_b)$) and $a \notin P$. We have $a \wedge b \wedge a \leq a$ and $a \wedge b \wedge a\,\theta_P\,a$ follows by Lemma 4.2. Hence $[a \wedge b \wedge a]_P \in \sigma_A(\mathcal{M}_b) \cap \mathcal{M}_a$.

To prove the converse, let $[a]_P \in \sigma_A(\mathcal{M}_b) \cap \mathcal{M}_a$. Hence $a \notin P$ and $b \notin P$. So $a \wedge b \wedge a \notin P$, $a \wedge b \wedge a \leq a$ and thus $a \wedge b \wedge a\,\theta_P\,a$ by Lemma 4.2.

To prove the second equality first assume that $[a\vee b\vee a]_P \in \mathcal{M}_{a\vee b\vee a}$. Hence $a \vee b \vee a \notin P$. If $a \notin P$ then $a\,\theta_P\,a \vee b \vee a$ by Lemma 4.2 because always $a \leq a \vee b \vee a$, and $[a \vee b \vee a]_P \in \mathcal{M}_a$ follows. If $a \in P$ then $b \notin P$ follows from $a \vee b \vee a \notin P$. Thus $a \vee b \vee a\,\theta_P\,0 \vee b \vee 0\,\theta_P\,b$ and $[a \vee b \vee a]_P \in \mathcal{M}_b \setminus \sigma_A(\mathcal{M}_a)$ follows.

Finally, assume that $[s]_P \in \mathcal{M}_a \cup (\mathcal{M}_b \setminus \sigma_A(\mathcal{M}_a))$. Then either $a \notin P$ and $[s]_P = [a]_P$, or $a \in P$, $b \notin P$ and $[s]_P = [b]_P$. In the first case it follows that

$a \vee b \vee a \notin P$ and thus $a \,\theta_P\, a \vee b \vee a$ by Lemma 4.2. Therefore $[s]_P = [a]_P \in \mathcal{M}_{a\vee b\vee a}$. In the second case it follows that $a \vee b \vee a \notin P$ and $a \vee b \vee a \theta_P 0 \vee b \vee 0 \theta_P b$. Again, $[s]_P = [b]_P \in \mathcal{M}_{a\vee b\vee a}$. $\square$

The attentive reader will point out that the étale map $q_A : \mathsf{Sk}(A) \to \mathsf{St}(A)$ cannot possibly be the topological dual of the skew algebra A because A gives the same étale map as its opposite algebra (in which the operations $\wedge$ and $\vee$ are the mirror version of those in A). Indeed, the étale map provides sufficient information only when A is right-handed (or left-handed), as will be shown in Section 4.3. We shall consider the general case in Section 4.4.

Let $f : A \to A'$ be a homomorphism of skew algebras. As we explained in Section 3, the induced homomorphism $A/\mathcal{D} \to A'/\mathcal{D}$ is dual to a proper partial map $f_\flat : \mathsf{St}(A') \to \mathsf{St}(A)$ with an open domain, which maps prime ideals to their inverse images, when defined. There is also a map $f_\sharp : \mathsf{Sk}(A') \to \mathsf{Sk}(A)$ of the same kind between the skew spectra. It is characterized by the requirement

$$f_\sharp([f(a)]_P) = [a]_{f_\flat(P)},$$

which uniquely determines the value of $f_\sharp$, when defined, because $a \,\theta_{f_\flat(P)}\, b$ is equivalent to $f(a) \,\theta_P\, f(b)$. If $f_\sharp([f(a)]_P)$ is defined then $f(a) \notin P$, hence $f_\sharp$ is defined on the basic copen set $\mathcal{M}_{f(a)}$, which shows that the domain of definition of $f_\sharp$ is open. Next, $f_\sharp$ is continuous and proper because $f_\sharp^{-1}(\mathcal{M}_a) = \mathcal{M}_{f(a)}$ for all $a \in A$.

The square

$$\begin{array}{ccc} \mathsf{Sk}(A) & \xleftarrow{\;f_\sharp\;} & \mathsf{Sk}(A') \\ {\scriptstyle q_A}\downarrow & & \downarrow{\scriptstyle q_{A'}} \\ \mathsf{St}(A) & \xleftarrow[\;f_\flat\;]{} & \mathsf{St}(A') \end{array} \tag{5}$$

commutes. It is easy to see that the values of $q_A \circ f_\sharp$ and $f_\flat \circ q_B$ coincide whenever they are both defined, so we only show that they have the same domain of definition. If $f_\flat(P)$ is defined then there is $a \in A$ such that $f(a) \notin P$, in which case $f_\sharp$ is defined at $[f(a)]_P$ and $q_B([f(a)]_P) = P$. Conversely, if $f_\sharp([f(a)]_P$ is defined then $f_\flat(q_B([f(a)]_P)) = f_\flat(P)$ is defined because $f(a) \notin P$.

Lemma 4.9 *The map $f_\sharp$ is a bijection on fibers, i.e., given any $P \in \mathrm{dom}(f_\flat)$, $f_\sharp$ maps $\mathsf{Sk}(B)_P \cap \mathrm{dom}(f_\sharp)$ bijectively onto the fiber $\mathsf{Sk}(A)_{f_\flat(P)}$.*

Proof. Consider any $P \in \mathrm{dom}(f_\flat)$. The Lemma states that $f_\sharp$ is a bijective map between the sets $\{[f(a)]_P \mid f(a) \notin P\}$ and $\{[a]_{f_\flat(P)} \mid a \notin f_\flat(P)\}$.

For injectivity, suppose $f_\sharp([f(a)]_P) = f_\sharp([f(a')]_P)$ where $f(a) \notin P$ and $f(a') \notin P$. By the definition of $f_\sharp$ it follows that $[a]_{f_\flat(P)} = [a']_{f_\flat(P)}$, which is equivalent to $a \,\theta_{f_\flat(P)}\, a'$, and hence $f(a) \,\theta_P\, f(a')$. This establishes the fact that $[f(a)]_P = [f(a')]_P$.

For surjectivity, pick any $[a]_{f_\flat(P)}$ where $a \notin f_\flat(P)$. It follows that $f(a) \notin P$, so $f_\sharp$ is defined at $[f(a)]_P$ and maps it to $[a]_{f_\flat(P)}$. $\square$

In view of the previous lemma one might contemplate turning $f_\sharp$ around, so that instead of a partial map which is bijective on fibers we would get a total map which is injective on fibers. The trouble is that the inverted map need not be continuous, so the topological nature of $f_\sharp$ must be obscured by a more complex condition.

4.2 From spaces to algebras

As we already indicated, the original algebra A cannot always be reconstructed from $q_A : \mathsf{Sk}(A) \to \mathsf{St}(A)$. However, if A is *right-handed* then $q_A : \mathsf{Sk}(A) \to \mathsf{St}(A)$ carries all the information needed, so we consider this case first.

We call a surjective étale map $p : E \to B$ between Boolean spaces a *skew Boolean space*. The corresponding right-handed skew algebra A_p consists of copen sections of p, i.e., an element of A_p is a copen subset $S \subseteq E$ such that $p|_S$ is injective. To describe the right-handed skew structure on A_p, recall the saturation operation $\sigma_p(S) = p^{-1}(p(S))$, and define for $S, R \in A_p$

$$\begin{aligned} 0 &= \emptyset, \\ S \wedge R &= \sigma_p(S) \cap R, \\ S \vee R &= S \cup (R - \sigma_p(S)), \\ S \setminus R &= S - \sigma_p(R), \\ S \cap R &= S \cap R. \end{aligned}$$

It is clear that these operations map back into A_p. For example, $S \cap R$ is a section, and it is copen because it is the intersection of two copen subsets of the Hausdorff space E.

It is not difficult to check that the above operations form a skew algebra with bare hands. An alternative, more elegant way of establishing the skew structure is to view the elements of A_p as partial maps $s, r : B \rightharpoonup E$ and exhibit A_p as a subalgebra of the right-handed skew algebra $\mathcal{P}(B, E)$ as described in Section 2.1.

The construction of the skew algebra A_p induces the usual construction of a generalized Boolean algebra via the lattice reflection, as follows.

Proposition 4.10 *Let $p : E \to B$ be a skew Boolean space and let B^* be the Boolean algebra of copen subsets of B. The map $A_p \to B^*$ defined by $S \mapsto p(S)$ is the lattice reflection of A_p.*

Proof. By Proposition 4.6 the map $S \mapsto p(S)$ is surjective, and it is easily seen to be a lattice homomorphism. Thus we only have to check $p(S) = p(R)$ is equivalent to $S \,\mathcal{D}\, R$, where S and R are copen sections of p. This follows from

$$\begin{aligned} S \preceq R \iff S \wedge R \wedge S = S \iff \sigma_p(S) \cap \sigma_p(R) \cap S = S \\ \iff S \subseteq \sigma_p(R) \iff p(S) \subseteq p(R). \end{aligned}$$

□

We turn attention to morphisms next. We already know that a homomorphism $f : A \to A'$ of skew algebras induces a commutative square (5) in which the horizontal partial maps are proper, continuous and have open domains of definition, and that the top map is bijective on fibers by Lemma 4.9. So we define a morphism (g, h) between skew Boolean spaces $p : E \to B$ and $p' : E' \to B'$ to be a commutative diagram

$$\begin{array}{ccc} E & \xrightarrow{g} & E' \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p'} \\ B & \xrightarrow[h]{} & B' \end{array} \tag{6}$$

in which g and h are proper continuous partial maps with open domains of definition. Furthermore, we require that g is a bijection on fibers, in the sense that g maps $E_x \cap \mathrm{dom}(g)$ bijectively onto $E_{h(x)}$ for every $x \in \mathrm{dom}(h)$.

Lemma 4.11 *If (g, h) is a morphism between skew Boolean spaces $p : E \to B$ and $p' : E' \to B'$ then $p(g^{-1}(S)) = h^{-1}(p'(S))$ for every copen section S in E'.*

Proof. Let S be a copen section in E'. If $y \in p(g^{-1}(S))$ then there exists $x \in g^{-1}(S)$ such that $y = p(x)$. Then $h(y) = h(p(x)) = p'(g(x)) \in p'(S)$, so that $y \in h^{-1}(p'(S))$ and $p(g^{-1}(S)) \subseteq h^{-1}(p'(S))$ follows. On the other hand, if $h(y) \in p'(S)$ then $h(y) = p'(z)$ for some $z \in S$. Because g is surjective on fibers there exists $x \in E_y$ with the property $g(x) = z$. Now, $y = p(x) \in p(g^{-1}(S))$ and we get $h^{-1}(p'(S)) \subseteq p(g^{-1}(S))$. □

We would like to construct a corresponding homomorphism $(g, h)_\natural : A_{p'} \to A_p$. For a copen section $S \subseteq E'$, the inverse image $g^{-1}(S)$ is copen in E because g is continuous and proper, and it is a section because g is injective on fibers. Thus we may define $(g, h)_\natural(S) = g^{-1}(S)$.

Let us show that g^{-1} commutes with the saturation operations σ_p and $\sigma_{p'}$. If S is a copen section in E', then $p(g^{-1}(S)) = h^{-1}(p'(S))$ by Lemma 4.11. Therefore

$$g^{-1}(\sigma_{p'}(S)) = p^{-1}(h^{-1}(p'(S)) = p^{-1}(p(g^{-1}(S)) = \sigma_p(g^{-1}(S)),$$

as claimed. The operations on A_p and $A_{p'}$ are defined in terms of basic set-theoretic operations and the saturation maps σ_p and σ'_p. Because g^{-1} commutes with all of them, $(g, h)_\natural$ is an algebra homomorphism.

The following is the counterpart of Proposition 4.10 for morphisms.

Proposition 4.12 *Let (g, h) be a morphism between skew Boolean spaces $p : E \to B$ and $p' : E' \to B'$. Its lattice reflection $(g, h)_\natural/\mathcal{D} : A_{p'}/\mathcal{D} \to A_p/\mathcal{D}$ is isomorphic to the lattice homomorphism $h^* : (B')^* \to B^*$ defined by $h^*(S) = h^{-1}(S)$.*

Proof. By Proposition 4.10 the vertical maps in the diagram

$$\begin{array}{ccc} A_p & \xleftarrow{(g,h)_\natural} & A_{p'} \\ \downarrow & & \downarrow \\ B^* & \xleftarrow[h^*]{} & (B')^* \end{array}$$

are lattice reflections. Because the right-hand vertical arrow is epi it suffices to show that the diagram commutes, which is just Lemma 4.11. ☐

4.3 Duality for right-handed algebras

Let us take stock of what we have done so far, in terms of category theory. On the algebraic side we have the category SkAlg of skew algebras and homomorphisms, as well as its reflective full subcategory $\mathsf{SkAlg}_\mathsf{R}$ on right-handed skew algebras.

On the topological side we have the category SkSp of skew Boolean spaces. A morphism $(g,h) : p \to p'$ between skew Boolean spaces $p : E \to B$ and $p' : E' \to B'$ is a commutative square (6) with proper continuous partial maps g and h whose domains of definition are open, and the top map g is bijective on fibers. Admittedly, the morphisms in SkSp are not very nice, but in Section 5 we show that they decompose nicely into partial identities and pullbacks.

In Section 4.1 we defined a functor

$$\mathcal{S} : \mathsf{SkAlg}^{\mathrm{op}} \to \mathsf{SkSp}$$

which assigns to each skew algebra A a skew Boolean space $\mathcal{S}(A) = q_A : \mathsf{Sk}(A) \to \mathsf{St}(A)$. The functor takes a homomorphism $f : A \to B$ to the corresponding morphism $\mathcal{S}(f) = (f_\sharp, f_\flat) : \mathcal{S}(B) \to \mathcal{S}(A)$.

In Section 4.2 we defined a functor

$$\mathcal{A} : \mathsf{SkSp}^{\mathrm{op}} \to \mathsf{SkAlg}_\mathsf{R}$$

which maps an étale map $p : E \to B$ between Boolean spaces to a right-handed skew algebra $\mathcal{A}(p) = A_p$, and a morphism $(g,h) : p \to p'$ as in (6) to a homomorphism $\mathcal{A}(g,h) = (g,h)_\natural : \mathcal{A}(p') \to \mathcal{A}(p)$.

We now work towards showing that $\mathcal{S}$ restricted to $\mathsf{SkAlg}_\mathsf{R}$ and $\mathcal{A}$ form a duality. For a skew algebra A define the map $\phi_A : A \to \mathcal{A}(\mathcal{S}(A))$ by

$$\phi_A(a) = \mathcal{M}_a.$$

That ϕ_A is an isomorphism of right-handed skew algebras follows from Lemma 4.3 and Proposition 4.8. By the lemma ϕ_A preserves $\cap$, it obviously preserves 0, and by the proposition it is a bijection which preserves the right-handed skew operations $\wedge$ and $\vee$:

$$\begin{aligned}\mathcal{M}_{a\wedge b} &= \mathcal{M}_{b\wedge a\wedge b} = \sigma_A(\mathcal{M}_a) \cap \mathcal{M}_b = \mathcal{M}_a \wedge \mathcal{M}_b \\ \mathcal{M}_{a\vee b} &= \mathcal{M}_{a\vee b\vee a} = \mathcal{M}_a \cup (\mathcal{M}_b \setminus \sigma_A(\mathcal{M}_a)) = \mathcal{M}_a \vee \mathcal{M}_b.\end{aligned}$$

Naturality of ϕ amounts to the identity

$$f_\sharp^{-1}(\mathcal{M}_a) = \mathcal{M}_{f(a)},$$

where $f : A \to B$ is a homomorphism and $a \in A$. After unraveling the definition of $f_\sharp$ we see that the set on the left-hand side consists of elements $[f(a)]_P$ with $f(a) \notin P$, which is just the description of the right-hand side. We have shown that ϕ is a natural isomorphism between the identity and $\mathcal{A} \circ \mathcal{S}$.

To establish the equivalence we also need an isomorphism ψ_p between a skew Boolean space $p : E \to B$ and $q_A : \mathsf{Sk}(A_p) \to \mathsf{St}(A_p)$, natural in p. It consists of two homeomorphism $(\psi_p)_\flat = h$ and $(\psi_p)_\sharp = g$ for which the following diagram commutes:

$$\begin{array}{ccc} E & \xrightarrow{\;g\;} & \mathsf{Sk}(A_p) \\ {\scriptstyle p}\Big\downarrow & & \Big\downarrow{\scriptstyle q_{A_p}} \\ B & \xrightarrow[\;h\;]{} & \mathsf{St}(A_p) \end{array} \tag{7}$$

Because the top map determines the bottom one, we consider g first. By Proposition 4.8 the map $S \mapsto \mathcal{M}_S$ is an order isomorphism between A_p and the copen sections of $\mathsf{Sk}(A_p)$. But since A_p is the set of copen sections of E, and the natural partial order in A_p coincides with the subset relation in E, the map $S \mapsto \mathcal{M}_S$ maps the basis for E isomorphically onto the basis for $\mathsf{Sk}(A_p)$. Consequently, the topologies of E and $\mathsf{Sk}(A_p)$ are isomorphic as posets, too, and because E and $\mathsf{Sk}(A_p)$ are sober spaces they are homeomorphic. Explicitly, the homeomorphism $g : E \to \mathsf{Sk}(A_p)$ induced by the isomorphism $S \mapsto \mathcal{M}_S$ takes a point $y \in E$ to the unique point $g(y) \in \mathsf{Sk}(A_p)$ satisfying, for all copen sections S in E,

$$y \in S \iff g(y) \in \mathcal{M}_S.$$

Similarly, the homeomorphism $h : B \to \mathsf{St}(A_p)$ is characterized by the requirement, for all copen sections S in E,

$$x \in p(S) \iff h(x) \in \mathcal{N}_S.$$

It is not hard to verify that $h(x) = \{\mathcal{D}_R \mid R \in A_p \wedge x \in p(R)\}$ and that $g(x) = [S]_{h(x)}$ for any $S \in A_p$ such that $x \in p(S)$.

We verify that (7) commutes by checking that the corresponding square of inverse image maps does. On one hand, starting with a copen section $\mathcal{N}_R$ in $\mathsf{St}(A_p)$, we have

$$p^{-1}(h^{-1}(\mathcal{N}_R)) = p^{-1}(R) = \bigcup\{S \mid p(S) \subseteq R\},$$

where S in the union ranges over copen sections in E. On the other hand,

$$g^{-1}(q_{A_p}^{-1}(\mathcal{N}_R)) = g^{-1}(\bigcup\{\mathcal{M}_S \mid p(S) \subseteq R\}) = \\ \bigcup\{g^{-1}(\mathcal{M}_S) \mid p(S) \subseteq R\} = \bigcup\{S \mid p(S) \subseteq R\},$$

where S again ranges over copen sections in E.

Naturality of ψ involves the commutativity of a cube which we prefer not to draw because six of its faces commute by definition and the two remaining faces are

$$\begin{array}{ccc} B & \xrightarrow{(\psi_p)_\flat} & \mathsf{St}(A_p) \\ {\scriptstyle h}\Big\downarrow & & \Big\downarrow{\scriptstyle (g,h)_{\natural\flat}} \\ B' & \xrightarrow[(\psi_{p'})_\flat]{} & \mathsf{St}(A_{p'}) \end{array} \qquad \begin{array}{ccc} E & \xrightarrow{(\psi_p)_\sharp} & \mathsf{Sk}(A_p) \\ {\scriptstyle g}\Big\downarrow & & \Big\downarrow{\scriptstyle ((g,h)_\natural)_\sharp} \\ E' & \xrightarrow[(\psi_{p'})_\sharp]{} & \mathsf{Sk}(A_{p'}) \end{array}$$

where $p : E \to B$ and $p' : E' \to B'$ are skew Boolean spaces and (g, h) is a morphism from p to p'. We check commutativity of the right-hand square, the other one is similar. Again, we verify that the corresponding square of inverse image maps commutes. For any copen section $\mathcal{M}_{S'}$ in the lower-right corner we have

$$g^{-1}((\psi_{p'})_\sharp^{-1}(\mathcal{M}_{S'})) = g^{-1}(S')$$

and

$$(\psi_p)_\sharp^{-1}(((g,h)_\natural)_\sharp^{-1}(\mathcal{M}_{S'})) = (\psi_p)_\sharp^{-1}(\mathcal{M}_{(g,h)_\natural(S')}) = (g,h)_\natural(S') = g^{-1}(S').$$

We have proved the following main theorem.

Theorem 4.13 *The category of right-handed skew Boolean algebras with intersections is dual to the category of skew Boolean spaces.*

Clearly, there is also duality between *left*-handed skew algebras and skew Boolean spaces, simply because the categories of left-handed and the right-handed skew algebras are isomorphic.

4.4 Duality for skew algebras

To see what is needed for duality in the case of a general skew algebra, consider what happens when we take a skew algebra A to its skew Boolean space $q_A : \mathsf{Sk}(A) \to \mathsf{St}(A)$, and then the space to the right-handed skew algebra A_{q_A}. By Proposition 4.8 the elements and the natural partial order do not change (up to isomorphism), but the operations do. The new ones are expressed in terms of the original ones as

$$x \wedge' y = y \wedge x \wedge y \qquad \text{and} \qquad x \vee' y = x \vee y \vee x.$$

If we want to recover $\wedge$ and $\vee$ from $\wedge'$ and $\vee'$ we need to break the symmetry that is present in $\wedge'$ and $\vee'$ by keeping around enough information about the original operations of A.

Recall that a rectangular band is a set with an operation which is idempotent, associative and it satisfies the rectangular identity. We can similarly define rectangular bands in any category with finite products. For example, a rectangular band in the category of étale maps over a given base space B is an étale map $p : E \to B$ together with a continuous map $\wedge : E \times_B E \to E$ over B which satisfies the required identities fiber-wise.

The skew Boolean space $q_A : \mathsf{Sk}(A) \to \mathsf{St}(A)$ carries the structure of a rectangular band whose operation $\curlywedge : \mathsf{Sk}(A) \times_{\mathsf{St}(A)} \mathsf{Sk}(A) \to \mathsf{Sk}(A)$ is defined by

$$[a]_P \curlywedge [b]_P = [a \wedge b]_P.$$

Idempotency and associativity of $\curlywedge$ follow immediately from the corresponding properties of $\wedge$ and the fact that θ_P is a congruence. To see that the rectangular identity is satisfied, let $a, b, c \notin P$. Because $(A, \wedge)$ forms a normal band, namely it satisfies the identity $x \wedge y \wedge z \wedge w = x \wedge z \wedge y \wedge w$, it follows that $a \wedge b \wedge c \leq a \wedge c$ and so $a \wedge b \wedge c \, \theta_P \, a \wedge c$ by Lemma 4.2, which is equivalent to $[a]_P \curlywedge [b]_P \curlywedge [c]_P = [a]_P \curlywedge [c]_P$.

We have to check that $\curlywedge$ is continuous. Let $([a]_P, [b]_P)$ be any point of the domain of $\curlywedge$ and suppose $[a \wedge b]_P \in \mathcal{M}_c$ for some $c \in A$. We seek an open neighborhood of $([a]_P, [b]_P)$ which is mapped into $\mathcal{M}_c$ by $\curlywedge$. Because basic open subsets of $\mathsf{Sk}(A) \times_{\mathsf{St}(A)} \mathsf{Sk}(A)$ are of the form

$$\{([u]_Q, [v]_Q) \mid Q \subseteq A \text{ prime ideal, } u \notin Q \text{ and } v \notin Q\},$$

it suffices to find $u, v \in A$ such that $a\,\theta_P\,u$ and $b\,\theta_P\,v$, and for all prime ideals $Q \subseteq A$, if $u \notin Q$ and $v \notin Q$ then $u \wedge v\,\theta_Q\,c$. We claim that

$$u = (a \wedge b \wedge a) \cap (c \wedge a)$$
$$v = (b \wedge a \wedge b) \cap (b \wedge c)$$

satisfy these conditions. We note that $u \wedge v \leq c$ because

$$u \wedge v = ((a \wedge b \wedge a) \cap (c \wedge a)) \wedge ((b \wedge a \wedge b) \cap (b \wedge c)) \leq (c \wedge a) \wedge (b \wedge c) \leq c,$$

where we used the fact that $\wedge$ is compatible with the natural partial order, which is the case because $(A, \wedge)$ is a normal band. Next observe that $a \wedge b \wedge a\,\theta_P\,c \wedge a$ because $[a \wedge b]_P \in \mathcal{M}_c$, hence

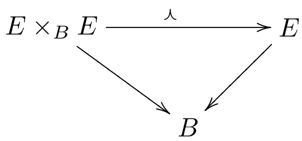
$$u = (a \wedge b \wedge a) \cap (c \wedge a)\,\theta_P\,(a \wedge b \wedge a) \cap (a \wedge b \wedge a) = a \wedge b \wedge a\,\theta_P\,a,$$

where the last step follows from Lemma 4.2 and $a, b \notin P$. We similarly show that $v\,\theta_P\,b$. If $Q \subseteq A$ is a prime ideal such that $u \notin Q$ and $v \notin Q$, then $u \wedge v \notin Q$ and since $u \wedge v \leq c$ we get $u \wedge v\,\theta_Q\,c$, again by Lemma 4.2.

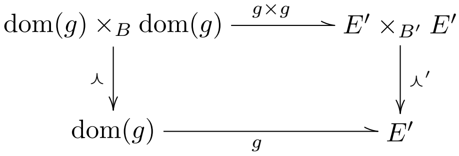
The rectangular band structure on A_p is precisely what is needed for duality in the general case.

Theorem 4.14 *The category of skew Boolean algebras with intersections is dual to the category of rectangular skew Boolean spaces.*

Proof. By rectangular skew Boolean space $(p : E \to B, \curlywedge)$ we mean a rectangular band in the category of surjective étale maps over B. More precisely, it is a skew Boolean space $p : E \to B$ together with a (not necessarily proper) continuous map $\curlywedge$ over B



which makes every fiber E_x into a rectangular band. Notice that in general a rectangular skew Boolean space is *not* a rectangular band in the category of skew Boolean spaces because $\curlywedge$ need not be proper. A morphism between $(p : E \to B, \curlywedge)$ and $(p' : E' \to B', \curlywedge')$ is a morphism of skew Boolean spaces $(g, h) : p \to p'$ which commutes with the operations on its domain of definition:



Note that the commutativity of the square implies that $\text{dom}(g)$ is closed under $\curlywedge$, so $\text{dom}(g) \cap E_x$ is a rectangular sub-band of E_x at every $x \in B$. And since g is bijective on fibers, $g|_x : \text{dom}(g) \cap E_x \to E_{h(x)}$ is an isomorphism of rectangular bands for every $x \in \text{dom}(h)$. We denote the category of rectangular skew Boolean spaces and their morphisms by SkRSp.

The duality is witnessed by a pair of contravariant functors

$$\mathcal{S} : \mathsf{SkAlg}^{\text{op}} \to \mathsf{SkRSp} \qquad \text{and} \qquad \mathcal{A} : \mathsf{SkRSp}^{\text{op}} \to \mathsf{SkAlg}.$$

The functor $\mathcal{S}$ maps a skew algebra A to the rectangular skew Boolean space $\mathcal{S}(A) = (p : \mathsf{Sk}(A) \to \mathsf{St}(A), \curlywedge)$, as described above. It takes a morphism $f : A \to A'$ to the morphism of skew Boolean spaces $\mathcal{S}(f) = (f_\sharp, f_\flat)$, which commutes with $\curlywedge$ because f commutes with $\wedge$.

The functor $\mathcal{A}$ maps a rectangular skew Boolean space $(p : E \to B, \curlywedge)$ to the skew algebra $\mathcal{A}(p, \curlywedge)$ whose elements are the copen sections of p and the operations are defined as follows:

$$\begin{aligned} 0 &= \emptyset, \\ S \wedge R &= (S \cap \sigma_p(R)) \curlywedge (\sigma_p(S) \cap R), \\ S \vee R &= (S - \sigma_p(R)) \cup (R - \sigma_p(S)) \cup (R \wedge S), \\ S \setminus R &= S - \sigma_p(R), \\ S \cap R &= S \cap R, \end{aligned}$$

These form a skew algebra because they are restrictions of the operations from Theorem 2.1.

The functor $\mathcal{A}$ maps a morphism $(g, h) : (p : E \to B, \curlywedge) \to (p' : E' \to B', \curlywedge')$ to the homomorphism $\mathcal{A}(g, h) = (g, h)_\natural$. We need to verify that $\mathcal{A}(g, h)$ preserves $\curlywedge$. Recall that $(g, h)_\natural$ is just g^{-1} acting on copen sections. In Section 4.2 we checked that g^{-1} commutes with the saturation operations. Because for every $x \in \text{dom}(h)$ the map $g|_x : E_x \cap \text{dom}(g) \to E'_{h(x)}$ is an isomorphism of rectangular bands, it is not hard to see that g^{-1} commutes with $\curlywedge$. Therefore, g^{-1} commutes with all the operations used to define the operations on $G(p, \curlywedge)$ and $G(p', \curlywedge')$, so it is a homomorphism of skew algebras.

It remains to be checked that $\mathcal{S} \circ \mathcal{A}$ and $\mathcal{A} \circ \mathcal{S}$ are naturally isomorphic to identity functors. Luckily, we can reuse a great deal of verification of duality for right-handed algebras from Section 4.3.

The natural isomorphism ϕ from the identity to $\mathcal{A} \circ \mathcal{S}$ is defined as in the right-handed case: for a skew algebra A set $\phi_A(a) = \mathcal{M}_a$. Thus we already know that it is a bijection which preserves intersections and relative complements, but we still have to check that it preserves meets and joins. It preserves meets because

$$\mathcal{M}_a \wedge \mathcal{M}_b = \mathcal{M}_{a \wedge b \wedge a} \curlywedge \mathcal{M}_{b \wedge a \wedge b} = \mathcal{M}_{a \wedge b}$$

where we used Proposition 4.8 in the first step and the fact that $[a \wedge b \wedge a]_P \curlywedge [b \wedge a \wedge b]_P = [a \wedge b]_P$ in the last step. With the help of Proposition 4.8 it is not hard to verify that whenever a and b commute then $\mathcal{M}_{a \vee b} = \mathcal{M}_a \cup \mathcal{M}_b = \mathcal{M}_{b \vee a}$, so ϕ_A preserves commuting joins. But since for arbitrary a and b their join can be expressed as a commuting join $a \vee b = (a \setminus b) \vee (b \setminus a) \vee (b \wedge a)$, and we already know that ϕ_A preserves $\setminus$ and $\wedge$, it follows that ϕ_A preserves joins. Naturality of ϕ_A is checked as in the right-handed case.

The natural isomorphism ψ from the identity to $\mathcal{S} \circ \mathcal{A}$ is defined as in the right-handed case. Given a rectangular skew Boolean space $(p : E \to B, \curlywedge)$, let $\psi_{p,\curlywedge}$ be the morphism consisting of the two homeomorphisms $(\psi_{p,\curlywedge})_\flat = h$ and $(\psi_{p,\curlywedge})_\sharp = g$ from diagram (7). All that we need to check in addition to what was already checked for ψ in Section 4.3 is that g preserves the rectangular band structure. For any $b \in B$, $x, y \in E_b$ and $T \in A_p$ we have

$$g(x \curlywedge y) \in \mathcal{M}_T \iff x \curlywedge y \in T.$$

On the other hand, if $g(x) = [S]_{h(b)}$ and $g(y) = [R]_{h(b)}$ then

$$g(x) \curlywedge g(y) \in \mathcal{M}_T \iff [S \curlywedge R]_{h(b)} \in \mathcal{M}_T \iff \\ S \curlywedge R \,\theta_{h(b)}\, T \text{ and } b \in p(T) \iff x \curlywedge y \in T.$$

We see that $g(x) \curlywedge g(y)$ and $g(x \curlywedge y)$ have the same neighborhoods, therefore they are equal. □

It may be argued that our duality has not gone all the way from algebra to geometry because a rectangular skew Boolean space still carries the algebraic structure of a rectangular band. However, this is not really an honest algebraic structure, as can be suspected from the fact that the category of non-empty rectangular bands is equivalent to the category of pairs of sets. The equivalence takes a rectangular band $(A, \wedge)$ to the pair of sets $(A/\mathcal{R}, A/\mathcal{L})$ where $A/\mathcal{R}$ and $A/\mathcal{L}$ are the quotients of A by Green's relations $\mathcal{R}$ and $\mathcal{L}$, respectively. In the other direction, a pair of sets (X, Y) is mapped to the rectangular band $X \times Y$ with the operation $(x_1, y_1) \wedge (x_2, y_2) = (x_1, y_2)$. The analogous decomposition of rectangular skew Boolean spaces yields the following variant of duality for skew algebras.

Theorem 4.15 *The category of skew Boolean algebras with intersections is dual to the category of pairs of skew Boolean spaces with common base.*

Proof. A pair of skew Boolean spaces with a common base is a diagram

$$E_L \xrightarrow{p_L} B \xleftarrow{p_R} E_R \tag{8}$$

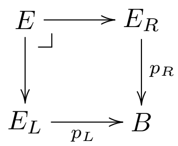
where $p_L : E_L \to B$ and $p_R : E_R \to B$ are skew Boolean spaces. A morphism is a commutative diagram

$$\begin{array}{ccccc} E_L & \xrightarrow{p_L} & B & \xleftarrow{p_R} & E_R \\ {\scriptstyle g_L}\downarrow & & {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle g_R} \\ E'_L & \xrightarrow[p'_L]{} & B' & \xleftarrow[p'_R]{} & E'_R \end{array} \tag{9}$$

in which the left- and right-hand square are morphisms of skew Boolean spaces (in the vertical direction). The diagrams are composed in the obvious way and we clearly get a category.

We establish the duality by showing that the category of pairs of skew Boolean spaces with common base is equivalent to the category of rectangular skew Boolean spaces. The idea is to have equivalence functors work at the level of fibers in the same way as the equivalence of non-empty rectangular bands and pairs of sets.

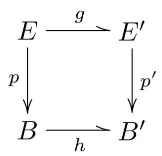
To convert a pair of skew Boolean spaces (8) into a rectangular Boolean space we form the pullback



to obtain a skew Boolean space $p : E \to B$. Concretely, the fiber E_x over $x \in B$ consists of pairs $(u, v) \in E_L \times E_R$ such that $p_L(u) = x = p_R(v)$, and $p(u, v) = p_L(u) = p_R(v)$. The rectangular band operation $\curlywedge$ on $p : E \to B$ defined by

$$(u_1, v_1) \curlywedge (u_2, v_2) = (u_1, v_2),$$

obviously makes the fibers into rectangular bands. A morphism (9) corresponds to the morphism

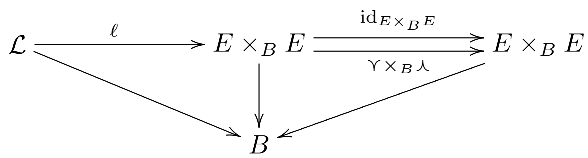


where g is the partial map with domain $\mathrm{dom}(g_L) \times_B \mathrm{dom}(g_R)$ defined by

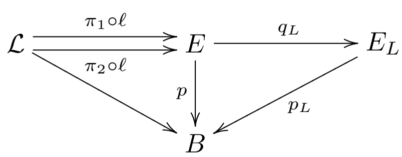
$$g(x, y) = (g_L(x), g_R(y)).$$

It clearly preserves $\curlywedge$.

In the opposite direction we start with a rectangular skew Boolean space $(p : E \to B, \curlywedge)$ and form a pair of skew Boolean spaces with a common base as follows. First construct the fiber-wise Green's relation $\mathcal{L}$ on E as the equalizer



in the topos of étale maps with base B. In the above diagram $\curlyvee : E \times_B E \to E$ is the operation associated with $\curlywedge$ by $x \curlyvee y = y \curlywedge x$. Still in the topos, we form the coequalizer



The quotient E_L is Hausdorff because by Proposition 2.2 the map q_L is open, and a pair of points in E may always be separated by clopen sections. We now have one of the skew Boolean spaces $p_L : E_L \to B$, and there is an analogous construction of $p_R : E_R \to B$. On a single fiber E_x over $x \in B$ the functor just performs the usual decomposition of the rectangular band E_x into its left- and right-handed factors $E_x/\mathcal{R}_x$ and $E_x/\mathcal{L}_x$. This is so because by Proposition 2.2 equalizers and coequalizers of étale maps are computed fiber-wise.

A morphism (g,h) from $(p : E \to B, \curlywedge)$ to $(p' : E' \to B', \curlywedge')$, displayed explicitly as inclusions of the domains of definition and total maps,

$$\begin{array}{ccccc} E & \longleftarrow & \mathrm{dom}(g) & \xrightarrow{g} & E' \\ {\scriptstyle p}\downarrow & & \downarrow & & \downarrow{\scriptstyle p'} \\ B & \longleftarrow & \mathrm{dom}(h) & \xrightarrow[h]{} & B' \end{array} \tag{10}$$

corresponds to a morphism between pairs of skew Boolean spaces as described next. Consider the commutative diagram

$$\begin{array}{ccccc} \mathcal{L} \cap (\mathrm{dom}(g) \times_B \mathrm{dom}(g)) & \rightrightarrows & \mathrm{dom}(g) & \xrightarrow{q_L} & q_L(\mathrm{dom}(g)) \\ & \searrow & {\scriptstyle p}\downarrow & & \downarrow \\ & & B & \xleftarrow[p_L]{} & E_L \end{array}$$

where the two parallel arrows into $\mathrm{dom}(g)$ are the restrictions of $\pi_1 \circ l$ and $\pi_2 \circ l$ to $\mathrm{dom}(g)$. By Proposition 2.2 the map q_L is open, hence $q_L(\mathrm{dom}(g))$ is an open subspace of E_L. Moreover, because $\mathrm{dom}(g)$ is an open subspace of E and q_L is open, it is not hard to check that the top row of the diagram is a coequalizer. Because g commutes with $\curlywedge$, the map $q'_L \circ g$ factors through the coequalizer,

$$\begin{array}{ccc} \mathrm{dom}(g) & \xrightarrow{g} & E' \\ {\scriptstyle q_L}\downarrow & & \downarrow{\scriptstyle q'_L} \\ q_L(\mathrm{dom}(g)) & \xrightarrow[g_L]{} & E'_L \end{array}$$

We have obtained a partial map $g_L : E_L \rightharpoonup E'_L$ whose domain $q_L(\mathrm{dom}(g))$ is an open subspace of E_L. Also q_L is proper because g is proper. We similarly obtain the right-handed version $g_R : E_R \rightharpoonup E'_R$. This gives us the desired morphism

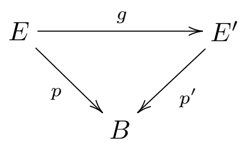
$$\begin{array}{ccccc} E_L & \xrightarrow{p_L} & B & \xleftarrow{p_R} & E_R \\ {\scriptstyle g_L}\downarrow & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle g_R} \\ E'_L & \xrightarrow[p'_L]{} & B' & \xleftarrow[p'_R]{} & E'_R \end{array}$$

To see that the two functors just described form an equivalence, we use the fact that fiber-wise they correspond to the equivalence between non-empty rectangular bands and pairs of non-empty sets. We omit the details. □

5 Variations

The morphisms between skew Boolean spaces were determined by our taking *all* homomorphisms on the algebraic side of duality. In this section we consider several variants in which the homomorphisms are restricted. We limit attention to the right-handed case, and ask the kind reader who will work out the general case to let us know whether there are any surprises. As a preparation we first show how morphisms of skew Boolean spaces decompose into open inclusions and pullbacks.

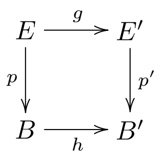
Lemma 5.1 *Suppose $p : E \to B$ and $p' : E' \to B$ are skew Boolean spaces and $g : E \to E'$ is a proper continuous map such that*



commutes. If g is bijective on fibers then it is a homeomorphism.

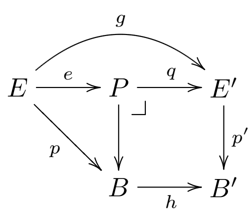
Proof. It is obvious that g is a bijection, so we only need to check that it is a closed map. If $K \subseteq E'$ is compact then the restriction $g|_{g^{-1}(K)} : g^{-1}(K) \to K$ is a closed map because it maps from the compact space $g^{-1}(K)$ to the Hausdorff space E'. Therefore, if $F \subseteq E$ is closed then $g(F) \cap K = g|_{g^{-1}(K)}(F \cap g^{-1}(K))$ is closed in K for every compact $K \subseteq E'$. Because E' is locally compact, it is compactly generated and we may conclude that $g(F)$ is closed. $\square$

Lemma 5.2 *Suppose $p : E \to B$ and $p' : E' \to B'$ are skew Boolean spaces and $g : E \to E'$ is a proper continuous map. A commutative square*



is a pullback if, and only if, g is bijective on fibers.

Proof. It is easy to check that g is bijective on fibers if the square is a pullback. Conversely, suppose g is bijective on fibers. We form the pullback of h and p' and obtain a factorization e, as in the diagram



The map e is proper because g is proper. Indeed, if $K \subseteq P$ is compact then $e^{-1}(K)$ is a closed subset of the compact subset $g^{-1}(q(K))$, therefore it is compact. Furthermore, e is bijective on fibers because g and q are. By Lemma 5.1 the map e is a homeomorphism, therefore the outer square is a pullback. $\square$

Consider a morphism of Boolean spaces, with inclusion of domains displayed explicitly:

$$
\begin{array}{ccccc}
E & \longleftarrow & \mathrm{dom}(g) & \xrightarrow{\;g\;} & E' \\
{\scriptstyle p}\big\downarrow & & \big\downarrow & & \big\downarrow{\scriptstyle p'} \\
B & \longleftarrow & \mathrm{dom}(h) & \xrightarrow[\;h\;]{} & B'
\end{array}
\tag{11}
$$

The left square need not be a morphism in our category because inclusions of open subsets need not be proper. If we turn them around they become *partial identities* with open domains of definitions, and we do get a decomposition

$$
\begin{array}{ccccc}
E & \rightharpoonup & \mathrm{dom}(g) & \xrightarrow{\;g\;} & E' \\
{\scriptstyle p}\big\downarrow & & \big\downarrow & & \big\downarrow{\scriptstyle p'} \\
B & \rightharpoonup & \mathrm{dom}(h) & \xrightarrow[\;h\;]{} & B'
\end{array}
\tag{12}
$$

in which both squares are morphisms between skew Boolean spaces. Now by Lemma 5.2 the condition that g is bijective on fibers is equivalent to the right square being a pullback. Therefore, every morphism can be decomposed into a partial identity (with open domain of definition) and a pullback. What does the decomposition (12) correspond to on the algebraic side of duality? In order to answer the question, we need to study a certain kind of ideals in skew algebras.

A $\leq$-*ideal* of a skew algebra A is a subset $I \subseteq A$ which is closed under finite joins and the natural partial order $\leq$. In particular, I is nonempty as it contains the empty join 0. A $\leq$-ideal may equivalently be described as a subalgebra that is closed under the natural partial order because in a skew algebra we always have $x \wedge y \leq y \vee x$. The following Lemma gives an explicit description of the $\leq$-ideal generated by a given subset, akin to how sets generate ideals in rings.

Lemma 5.3 *The $\leq$-ideal $\langle S\rangle_{\leq}$ generated by a subset $S \subseteq A$ is formed as the closure by finite joins of the downard closure of S with respect to the natural partial order:*

$$\langle S\rangle_{\leq} = \{x_1 \vee \cdots \vee x_n \mid \forall i \leq n \,.\, \exists y_i \in S \,.\, x_i \leq y_i\}.$$

Proof. Because any ideal that contains S also contains $\langle S\rangle_{\leq}$ we only have to check that $\langle S\rangle_{\leq}$ is a $\leq$-ideal. The set $\langle S\rangle_{\leq}$ is obviously closed under joins. To see that it is closed under the natural partial order, let $x \leq x_1 \vee \cdots \vee x_n$ where $x_i \leq y_i$ and $y_i \in S$. Then

$$x = (x_1 \vee \cdots \vee x_n) \wedge x \wedge (x_1 \vee \cdots \vee x_n) = \\ (x_1 \wedge x \wedge x_1) \vee \cdots \vee (x_n \wedge x \wedge x_n),$$

where we canceled all terms of the form $x_i \wedge x \wedge x_j$ with $i \neq j$, by the usual argument in the skew lattice theory that amounts to the fact that $(A, \vee)$ is *regular* as a band, i.e., it satisfies the identity $a \vee b \vee a \vee c \vee a = a \vee b \vee c \vee a$. Now, $x_i \wedge x \wedge x_i \leq x_i \leq y_i$ for all $i \leq n$ and thus $x \in \langle S\rangle_{\leq}$. □

Note that in the previous lemma the order of operations matters. If we first close under joins and then perform the downward closure we need not get a $\leq$-ideal because the resulting set need not be closed under joins.

We say that a subset $S \subseteq A$ of a skew algebra A is $\leq$-*cofinal* when $\langle S \rangle_{\leq} = A$. A homomorphism is $\leq$-cofinal when its image is $\leq$-cofinal. If A is commutative, $S \subseteq A$ is $\leq$-cofinal precisely when it is cofinal in the usual sense: for every $x \in A$ there is $y \in S$ such that $x \leq y$.

Every homomorphism $f : A \to A'$ of skew algebras may be decomposed into a $\leq$-cofinal morphism and an inclusion of a $\leq$-ideal

$$\begin{array}{ccc} A & \xrightarrow{\quad f \quad} & A' \\ & {\scriptstyle f}\searrow \quad \nearrow{\scriptstyle i} & \\ & \langle \mathrm{im}(f) \rangle_{\leq} & \end{array}$$

The following two propositions show that the decomposition is dual to the decomposition (12) on the topological side of duality.

Proposition 5.4 *Partial identities with open domains of definition on the topological side are dual to inclusions of $\leq$-ideals on the algebraic side.*

Proof. Let $p : E \to B$ be a skew Boolean space with open subsets $U \subseteq E$ and $V \subseteq B$ such that $p(U) = V$. These determine a morphism of skew Boolean spaces

$$\begin{array}{ccc} E & \xrightarrow{\;i\;} & U \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p|_U} \\ B & \xrightarrow[\;j\;]{} & V \end{array}$$

where i and j are the identity maps restricted to U and V, respectively. It is easy to check that the corresponding homomorphism $f = (i, j)_{\natural} : A_{p|_U} \to A_p$ is the inclusion of the subalgebra $A_{p|_U}$ into A_p. Its image is downward closed with respect to $\leq$ because the partial order in A_p is inclusion of copen sections.

Conversely, let $I \subseteq A$ be a $\leq$-ideal in A and $i : I \to A$ the inclusion. The dual of i is the morphism of Boolean spaces

$$\begin{array}{ccc} \mathsf{Sk}(A) & \xrightarrow{\;i_\sharp\;} & \mathsf{Sk}(I) \\ {\scriptstyle q_A}\downarrow & & \downarrow{\scriptstyle q_I} \\ \mathsf{St}(A) & \xrightarrow[\;i_\flat\;]{} & \mathsf{St}(I) \end{array}$$

where $i_\sharp$ acts as $i_\sharp([a]_P) = [a]_{P \cap I}$ and is defined for any $a \in I$ and a prime ideal $P \subseteq A$ such that $a \notin P$. The domain of $i_\sharp$ is open because it is the union of those basic copen sets $\mathcal{M}_a$ for which $a \in I$. The map $i_\sharp$ is open because it takes $\mathcal{M}_a \subseteq \mathsf{Sk}(A)$ to $\mathcal{M}_a \subseteq \mathsf{Sk}(I)$. Therefore, $i_\sharp$ really is (isomorphic to) a partial identity with an open domain of definition. The same fact for $i_\flat$ follows easily. $\square$

Proposition 5.5 *Pullbacks on the topological side are dual to $\leq$-cofinal homomorphisms on the algebraic side.*

Proof. Consider a morphism of skew Boolean spaces that is a pullback, which is equivalent to it being defined everywhere:

$$\begin{array}{ccc} E & \xrightarrow{\;g\;} & E' \\ {\scriptstyle p}\downarrow & \lrcorner & \downarrow{\scriptstyle p'} \\ B & \xrightarrow[\;h\;]{} & B' \end{array}$$

To see that the corresponding homomorphism $f = (g,h)_\natural : A_{p'} \to A_p$ is $\leq$-cofinal, let S be a copen section in E. Because g is everywhere defined and S is compact, there exist finitely many copen sections $R_1, \ldots, R_n$ in E' such that S is covered by the copen sections $g^{-1}(R_1), \ldots, g^{-1}(R_n)$. Let $T_1, \ldots, T_n$ be defined by $T_1 = g^{-1}(R_1) \cap S$ and $T_{i+1} = (g^{-1}(R_{i+1}) \cap S) \setminus T_i$ for $i \geq 1$. Then $S = T_1 \cup \ldots \cup T_n = T_1 \vee \ldots \vee T_n$ where the latter equlity follows because distinct T_i's have disjoint saturations. Furthermore, each T_i is contained in $g^{-1}(R_i) = f(R_i)$ and thus S lies in $\langle \mathrm{im}(f) \rangle_{\leq}$.

Next, let $f : A \to A'$ be a $\leq$-cofinal homomorphism of right-handed skew algebras. We claim that $f_\sharp : \mathsf{Sk}(A') \to \mathsf{Sk}(A)$ is everywhere defined. To see this take any $[b]_P \in \mathsf{Sk}(A')$. Since f is $\leq$-cofinal there exist $b_1, \ldots, b_n \in A'$ and $a_1, \ldots, a_n \in A$ such that $b = b_1 \vee \ldots \vee b_n$ and $b_i \leq f(a_i)$ for all $i = 1, \ldots, n$. Since $b \notin P$ there exists i such that $b_i \notin P$. Thus $b_i \,\theta_P\, f(a_i)$ by Lemma 4.2. Let $\{i_1, \ldots i_k\}$ be the set of those indices i_j that satisfy $b_{i_j} \notin P$. Then $b \,\theta_P\, f(a_{i_1}) \vee \ldots \vee f(a_{i_k}) = f(a_{i_1} \vee \cdots \vee a_{i_k})$ and thus $[b]_P \in \mathrm{dom}(f_\sharp)$. □

Let us call a morphism of skew Boolean spaces

$$\begin{array}{ccc} E & \xrightarrow{\;g\;} & E' \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p'} \\ B & \xrightarrow[\;h\;]{} & B' \end{array}$$

total when both g and h are total, and *semitotal* when h is total. A morphism is total precisely when it is a pullback square. As direct consequences of Propositions 5.4 and 5.5 we obtain the dualities stated by the following pair of theorems.

Theorem 5.6 *The category of skew Boolean spaces and partial identities with open domains is dual to the category of right-handed skew algebras and inclusions of $\leq$-ideals.*

Theorem 5.7 *The category of skew Boolean spaces and total morphisms is dual to the category of right-handed skew algebras and $\leq$-cofinal homomorphisms.*

The duality for the semitotal morphisms is similar to duality for total morphisms, except that we have to replace $\leq$-cofinality with $\preceq$-cofinality: a subset $S \subseteq A$ of a skew algebra is *$\preceq$-cofinal* when the ideal generated by S equals A. Such an ideal may be computed either as the $\vee$-closure of $\preceq$-closure of S, or as the $\preceq$-closure of $\vee$-closure of S.

A homomorphism $f : A \to A'$ is $\preceq$-cofinal when its image is $\preceq$-cofinal. Because the image is closed under finite joins, $\preceq$-cofinality of f amounts to the following condition: for every $y \in A'$ there is $x \in A$ such that $y \preceq f(x)$. Cofinal homomorphisms between commutative algebras are also known as *proper* homomorphisms, but we avoid this terminology because we already use the term proper on the topological side. In a slightly different setup this kind of duality is considered by Ganna Kudryavtseva [4].

Theorem 5.8 *The category of skew Boolean spaces and semitotal morphisms is dual to the category of right-handed skew algebras and $\preceq$-cofinal homomorphisms.*

Proof. Assume that h in diagram (12) is total. For every $S \in B^*$ there exist $R_1, \ldots, R_n \in (B')^*$ such that S (as a copen set in B) is covered by the copen set $h^{-1}(R_1 \cup \ldots \cup R_n)$. Hence $S \leq h^*(R_1 \vee \ldots \vee R_n)$.

To prove the converse, assume that $f : A \to A'$ is a homomorphism of right-handed skew algebras that is cofinal with respect to $\preceq$. We claim that $f_\flat : \mathsf{St}(A') \to \mathsf{St}(A)$ is everywhere defined. To see this take any $P \in \mathsf{St}(A')$. Since f is $\preceq$-cofinal it follows that $\mathrm{im}(f)$ is not contained in any proper ideal of A'. Hence there exists $a \in A$ such that $f(a) \notin P$, which implies $f^{-1}(P) \neq A$. Thus $f^{-1}(P)$ is a prime ideal in A and so $f_\flat$ is defined at P, namely $f_\flat(P) = f^{-1}(P)$. □

To get still more variations of duality we consider notions of saturation. We say that a homomorphism $f : A \to A'$ of skew algebras is *$\mathcal{D}$-saturated* when its image is saturated with respect to Green's relation $\mathcal{D}$.

Lemma 5.9 *A homomorphism between skew algebras is $\mathcal{D}$-saturated if, and only if, it maps each $\mathcal{D}$-class surjectively onto a $\mathcal{D}$-class.*

Proof. The "if" part is obvious. For the "only if" part, let $f : A \to A'$ be a $\mathcal{D}$-saturated homomorphism. Suppose $a \in A$ and $b \in A'$ such that $f(a)\,\mathcal{D}\,b$. Because f is $\mathcal{D}$-saturated there is $a' \in A$ such that $f(a') = b$. Consider

$$a'' = (a' \wedge a \wedge a') \vee a \vee (a' \wedge a \wedge a').$$

It is obvious that $a''\mathcal{D}a$. Because $f(a)\mathcal{D}b$ we get $b \wedge f(a) \wedge b = b$ and $b \vee f(a) \vee b = b$, which implies

$$f(a'') = (b \wedge f(a) \wedge b) \vee f(a) \vee (b \wedge f(a) \wedge b) = b \vee f(a) \vee b = b,$$

as desired. □

There is a notion of saturation of morphisms on the topological side, too. Say that $(g, h) : p \to p'$ from $p : E \to B$ to $p' : E' \to B'$ is *saturated* when the domain of g is saturated with respect to p, i.e., $\mathrm{dom}(g) = p^{-1}(\mathrm{dom}(h))$. This is equivalent to the left square in (11) being a pullback. In a different context such morphisms were called partial pullbacks by Erik Palmgren and Steve Vickers [8].

The two notions of saturation are *not* dual to each other, so they yield two more dualities.

Theorem 5.10 *The category of right-handed skew algebras and $\mathcal{D}$-saturated homomorphisms is dual to the category of skew Boolean spaces and those morphisms*

$$\begin{array}{ccc} E & \xrightarrow{g} & E' \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p'} \\ B & \xrightarrow[h]{} & B' \end{array}$$

that satisfy the following lifting property: for every copen $U \subseteq B'$ and copen section S of p above $h^{-1}(U)$ there exists a copen section R of p' above U such that $S = g^{-1}(R)$.

Proof. Let $f : A \to A'$ be a $\mathcal{D}$-saturated homomorphism. To show that $(f_\sharp, f_\flat) : A_{p'} \to A_p$ has the desired property, consider a copen $\mathcal{N}_a \subseteq \mathsf{St}(A)$ and a copen section $\mathcal{M}_b \in \mathsf{Sk}(A')$ above $\mathcal{N}_{f(a)}$. Because $\mathcal{N}_{f(a)} = \mathcal{N}_b$, we have $f(a)\,\mathcal{D}\,b$ and so by Lemma 5.9 there exists $a' \in A$ such that $a'\;\mathcal{D}\;a$ and $b = f(a')$. So $\mathcal{N}_a = \mathcal{N}_{a'}$ and $f_\sharp^{-1}(\mathcal{M}_{a'}) = \mathcal{M}_{f(a')} = \mathcal{M}_b$.

Conversely, suppose we have a morphism between skew Boolean spaces, as in the statement of the theorem. We need to show that the corresponding homomorphism $f = (g, h)_\natural : A_{p'} \to A_p$ is $\mathcal{D}$-saturated. Let $S \subseteq E'$ be a copen section of p' above $V \subseteq B'$, and let $R \subseteq E$ be a copen section of p above $h^{-1}(V)$, i.e., $R\,\mathcal{D}\,g^{-1}(S) = f(S)$. By the property of our morphism there exists a copen section $S' \subseteq E'$ above V such that $f(S') = g^{-1}(S') = R$, as required. $\square$

Theorem 5.11 *The category of skew Boolean spaces and saturated morphisms is dual to the category of right-handed skew algebras and those homomorphisms $f : A \to A'$ for which $\langle \mathrm{im}(f) \rangle_\leq$ is closed under the natural preorder.*

Proof. Consider a saturated morphism

$$\begin{array}{ccc} E & \xrightarrow{g} & E' \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p'} \\ B & \xrightarrow[h]{} & B' \end{array}$$

and let $f = (g, h)_\natural : A_{p'} \to A_p$ be the corresponding homomorphism. Let R be a section of p' above a copen $V \subseteq B'$ and S a copen section of p above a copen $U \subseteq h^{-1}(V)$. Because $\mathrm{dom}(g)$ is saturated it contains S. For every $x \in S$ there is a copen section T of p' above V which passes through $g(x)$, hence x is covered by the copen section $g^{-1}(T)$. Because S is compact, there are finitely many sections $T_1, \ldots, T_n$ of p' above V such that each $x \in S$ is covered by some $g^{-1}(T_i)$. If we let $S_i = S \cap g^{-1}(T_i)$ then $S = S_1 \vee \cdots \vee S_n$ and $S_i \subseteq g^{-1}(T_i)$. We have proved that $\langle \mathrm{im}(f) \rangle_\leq$ is closed under $\preceq$.

Conversely, consider a homomorphism $f : A \to A'$ such that $\langle \mathrm{im}(f) \rangle_\leq$ is closed under $\preceq$. Suppose $P \subseteq A'$ is a prime ideal such that $f^{-1}(P)$ is also a prime ideal. Given any $a' \in A' - P$, we need to show that $f_\sharp$ is defined at $[a']_P$. There exists $a \in A$ such that $f(a) \notin P$. By Lemma 5.12, proved below, there is $b \in A' - P$ such that $a'\;\theta_P\;b$ and $b\;\mathcal{D}\;f(a)$. It follows that $b \preceq f(a)$

and thus $b \in \langle \mathrm{im}(f) \rangle_{\leq}$ by the assumption. Hence there exits $a_1, \ldots, a_n \in A$ and $b_1, \ldots, b_n \in A'$ such that $b = b_1 \vee \cdots \vee b_n$ and $b_i \leq f(a_i)$ for all i. Because $b = b_1 \vee \cdots \vee b_n$ and $b \notin P$ it follows that not all b_i can lie in P. Say $b_1, \ldots, b_j \notin P$ while $b_{j+1}, \ldots, b_n \in P$. Then $b \, \theta_P \, b_1 \vee \cdots \vee b_j \, \theta_P \, f(a_1 \vee \cdots \vee a_j)$ by Lemma 4.2 and so $f_\sharp([a']_P) = f_\sharp([b]_P) = [a_1 \vee \cdots \vee a_j]_{f_\flat(P)}$ is defined. □

Lemma 5.12 *Let A be a skew algebra and $P \subseteq A$ a prime ideal. For every $a, b \in A - P$ there is $c \in A - P$ such that $a \, \theta_P \, c$ and $c \, \mathcal{D} \, b$.*

Proof. If we take $c = (a \wedge b \wedge a) \vee b \vee (a \wedge b \wedge a)$ then $c \, \mathcal{D} \, b$ obviously holds. Next, $a \notin P$ and $b \notin P$ together imply $a \wedge b \wedge a \notin P$, and $a \wedge b \wedge a \, \theta_P \, a$ follows by Lemma 4.2. Finally, $a \wedge b \wedge a \leq c$ and $a \wedge b \wedge a \, \theta_P \, c$ follows, again by Lemma 4.2.

□

6 Lattice sections of skew algebras

A *lattice section* of a skew lattice A is a section $\ell : A/\mathcal{D} \to A$ of the canonical projection $q : A \to A/\mathcal{D}$ which preserves 0, $\wedge$ and $\vee$. We construct a right-handed skew Boolean algebra without a lattice section. This answers negatively the open question whether every skew lattice has a section.

Proposition 6.1 *A right-handed skew algebra has a lattice section if, and only if, the corresponding skew Boolean space has a global section.*

Proof. Let A be a skew algebra with a lattice section $\ell : A/\mathcal{D} \to A$. For every $d \in A/\mathcal{D}$ the étale map $q_A : \mathsf{Sk}(A) \to \mathsf{St}(A)$ has a local section $s_d : \mathcal{N}_d \to E$ which maps $\mathcal{N}_d$ to $\mathcal{M}_{\ell(d)}$. We can glue the local sections into a global one as long as they are compatible. To see that this is the case, take any $d, e \in A/\mathcal{D}$ and compute by Lemma 4.8

$$\mathcal{M}_{\ell(d \wedge e)} = \mathcal{M}_{\ell(e \wedge d \wedge e)} = \mathcal{M}_{\ell(e) \wedge \ell(d) \wedge \ell(e)} = q_A^{-1}(\mathcal{N}_d) \cap \mathcal{M}_{\ell(e)} = q_A^{-1}(\mathcal{N}_d \cap \mathcal{N}_e) \cap \mathcal{M}_{\ell(e)},$$

and similarly

$$\mathcal{M}_{\ell(e \wedge d)} = \mathcal{M}_{\ell(d \wedge e \wedge d)} = q_A^{-1}(\mathcal{N}_d \cap \mathcal{N}_e) \cap \mathcal{M}_{\ell(d)}.$$

Therefore, s_d and s_e restricted to $\mathcal{N}_d \cap \mathcal{N}_e$ are both equal to $s_{d \wedge e}$.

Conversely, suppose $p : E \to B$ is a skew Boolean space with a global section $s : B \to E$. Then the map $V \mapsto s(V)$ is a lattice section for A_p, since $s(\emptyset) = \emptyset$,

$$s(U) \wedge s(V) = p^{-1}(p(s(U))) \cap s(V) = p^{-1}(U) \cap s(V) = s(U) \cap s(V) = s(U \cap V),$$

and similarly $s(U) \vee s(V) = s(U \cup V)$. □

The preceeding construction of a global section from a lattice section $\ell : A/\mathcal{D} \to A$ used only preservation of 0 and $\wedge$ by ℓ. Thus we proved in passing that a right-handed skew algebra has a lattice section if, and only if, it has a section which preserves 0 and $\wedge$.

We next construct a skew Boolean space which does not have a global section. Consequently, the corresponding skew algebra does not have a lattice section. To find a counter-example we first look at a sufficient condition for existence of global sections.

Proposition 6.2 *A skew Boolean space has a global section if the base space is a countable union of compact open sets. Dually, a right-handed skew algebra has a lattice section if it contains a cofinal countable chain for to the natural preorder.*

Proof. Suppose the base B of a skew Boolean space $p : E \to B$ of Boolean spaces is covered by countably many copen sets. Because a finite union of such sets is again copen, there is in fact a countable chain $C_0 \subseteq C_1 \subseteq \cdots$ of copen sets which cover B. Each C_i has a local section $s_i : C_i \to E$. These can be used to form a global section $s : B \to E$ which equals s_i on $C_i - C_{i-1}$.

To transfer the construction to the algebraic side of duality, observe that a sequence $c_0, c_1, c_2, \ldots$ in A is a cofinal chain for the natural preorder if, and only if, the corresponding sequence of copen sets $\mathcal{N}_{c_0}, \mathcal{N}_{c_1}, \mathcal{N}_{c_2}, \ldots$ is a chain that covers $\mathsf{St}(A)$. $\square$

Thus we must look for a counter-example whose base space is fairly large. A simple one is the first uncountable ordinal ω_1 with the interval topology. Recall that the points of ω_1 are the countable ordinals and that the interval topology is generated by the open intervals $I_{\alpha,\beta} = \{\gamma \in \omega_1 \mid \alpha < \gamma < \beta\}$ for $\alpha < \beta < \omega_1$. We also need a total space, for which we take the disjoint sum of intervals

$$E = \coprod_{\alpha<\omega_1} [0, \alpha],$$

where each interval $[0, \alpha]$ carries the interval topology. The points of E are pairs (α, β) with $\beta \leq \alpha$ and the basic open sets are of the form $\{\alpha\} \times I_{\beta,\gamma}$ for $\beta < \gamma \leq \alpha$. We define the étale map $p : E \to \omega_1$ to be the second projection, so that the fiber above β contains points (α, β) with $\beta \leq \alpha < \omega_1$.

A global section $\omega_1 \to E$ of p is a map $\beta \mapsto (s(\beta), \beta)$ where $s : \omega_1 \to \omega_1$ is a progressive locally constant map. By progressive we mean that $\beta \leq s(\beta)$ for all $\beta < \omega_1$. To see that s is locally constant, observe that for any $\beta \in \omega$ the preimage of the open subset $\{s(\beta)\} \times [0, s(\beta)] \subseteq E$ under the section is open, hence β has an open neighborhood which s maps to $\{s(\beta)\}$.

The map s preserves suprema of monotone sequences because it is locally constant. Above every $\beta < \omega_1$ there is a fixed point of s, namely the supremum of the monotone sequence

$$\beta \leq s(\beta) \leq s(s(\beta)) \leq \cdots$$

By using this fact repeatedly, we obtain a strictly increasing sequence $\gamma_0 < \gamma_1 < \gamma_2 < \cdots$ of fixed points of s. Their supremum $\gamma_\infty = \sup_i \gamma_i$ is a fixed point of s, too. Since s is locally constant there exists γ_i such that $s(\gamma_i) = s(\gamma)$, which yields the contradiction $\gamma_\infty = s(\gamma_\infty) = s(\gamma_i) = \gamma_i < \gamma_\infty$. We therefore conclude that s does not exist and $p : E \to \omega_1$ does not have a global section.

References

[1] R. J. Bignall and J. E. Leech. Skew boolean algebras and discriminator varieties. *Algebra Universalis*, 33:387–398, 1995.

[2] W. H. Cornish. Boolean skew algebras. *Acta Mathematica Academiae Scientiarum Hungarkcae Tomus*, 36:281–291, 1980.

[3] P. T. Johnstone. *Stone spaces*. Cambridge University Press, 1982.

[4] G. Kudryavtseva. A refinement of Stone duality to skew Boolean algebras. `http://arxiv.org/abs/1102.1242v2`, 2011.

[5] J. Leech. Skew lattices in rings. *Algebra Universalis*, 26:48–72, 1989.

[6] J. Leech. Normal skew lattices. *Semigroup forum*, 44:1–8, 1992.

[7] S. MacLane and I. Moerdijk. *Sheaves in Geometry and Logic: A First Introduction to Topos Theory*. Springer-Verlag, 1992.

[8] E. Palmgren and S. J. Vickers. Partial Horn logic and cartesian categories. *Annals of Pure and Applied Logic*, 145(3):314–353, 2007.

[9] M. H. Stone. The theory of representations for Boolean algebras. *Transactions of the American Mathematical Society*, 40(1):37–111, July 1936.

[10] M. H. Stone. Applications of the theory of Boolean rings to general topology. *Transactions of the American Mathematical Society*, 41(3):375–481, May 1937.