# The Role of the Interval Domain in Modern Exact Real Airthmetic


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# Teaching theoreticians a lesson

Recently I have been told by an anonymous referee that

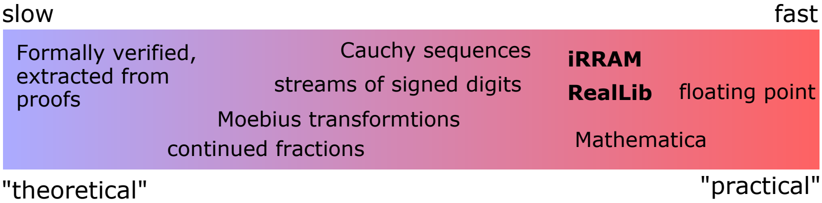
> *"Theoreticians do not like to be taught lessons."*

and by a friend that

> *"You should stop competing with programmers."*

In defiance of this advice, I shall talk about the lessons I learned, as a theoretician, in programming exact real arithmetic.

# The spectrum of real number computation




- Common features:
  - Reals are represented by successive approximations.
  - Approximations may be computed to any desired accuracy.
- State of the art, as far as speed is concerned:
  - iRRAM by Norbert Müller,
  - RealLib by Branimir Lambov.

# What makes iRRAM and ReaLib fast?

- Reals are represented by sequences of *dyadic* intervals (endpoints are rationals of the form $m/2^k$).
- The approximating sequences need *not* be nested chains of intervals.
- No guarantee on speed of converge, but arbitrarily fast convergence is possible.
- Previous approximations are *not* stored and *not* reused when the next approximation is computed.
- Each next approximation roughly doubles the amount of work done.

# The theory behind iRRAM and RealLib

- Theoretical models used to design iRRAM and RealLib:
  - Type Two Effectivity
  - a version of Real RAM machines
  - Type I representations
- The authors explicitly reject domain theory as a suitable computational model.
  - For example, because representing reals as nested chains of intervals is a bad idea.
  - They seem to have convinced even some high priests of domain theory.

# Our goal

- There is usually a significant gap between a theoretical model and the actual practical implementation.
- We wanted exact real arithmetic that was both efficient and easily formalizable:
  - Extract specification from the theory—automatically.
  - Leave the programmer freedom to produce fast implementation.
- We needed a tool that could extract specifications from formal descriptions of mathematical theories.

# Representations and realizability

- There are many kinds of representations:
  - Numbered sets (Eršov)
  - Type Two Representations (Weihrauch)
  - Domain representations (Blanck)

  These are useful in the *theory* of computability.
- Programmers use representations by *real-world programs*.
- In a related project, Chris Stone and Andrej Bauer built a tool *RZ* which translates first-order *constructive* theories into Objective Caml representations.
- RZ uses the realizability interpretation, and handles first-order logic, dependent types, and more.

# Construction of reals

- We axiomatized the following theories (constructively):
  - the ring of integers $\mathbb{Z}$,
  - the "approximate field" of dyadic rational numbers $\mathbb{D}$,
  - the poset of dyadic intervals $\mathbb{ID}$,
  - $\omega$-cpos as completions of their bases,
  - the interval domain $\mathbb{IR}$ as the completion of $\mathbb{ID}$,
  - the field of real numbers $\mathbb{R}$ as the maximal elements of $\mathbb{IR}$.
- We ran RZ on these to obtain specifications for Objective Caml (given as module signatures with logical assertions).
- We implemented the specifications by hand.
- We hope our implementation satisfies all the assertions …
- Logical reverse-engineering: which logical theory does RZ translate to a specification for exact real arithmetic in the style of iRRAM and RealLib?
- Initially, we did *not* expect domain theory to play a prominent role.

# Integers

- The integers are axiomatized as:
  - decidable ordered commutative ring with unit,
  - induction principle for natural numbers,
  - integer division,
  - shift operations $\mathsf{shl}_k(n) = 2^k \cdot n$ and $\mathsf{shr}_k(n) = \lfloor n/2^k \rfloor$,
  - binary logarithm $n \mapsto \lceil \log_2 n \rceil$.
- We used efficient implementations of large integer arithmetic:
  - Numerix by Gabriel Quercia
  - GNU Multiple Precision Library

# Dyadic rationals

- Dyadic rationals $\mathbb{D}$ form a decidable ordered ring.
- Only *approximate* division:

$$\forall k \in \mathbb{N}.\, \forall x, y \in \mathbb{D}.\, (y \neq 0 \implies \exists z \in \mathbb{D}.\, |x/y - z| < 2^{-k})$$

- In fact, we need approximate versions of all ring operations, e.g.:

$$\forall k \in \mathbb{N}.\, \forall x, y \in \mathbb{D}.\, \exists z \in \mathbb{D}.\, |(x + y) - z| < 2^{-k}$$

- This allows us to control the size of dyadic numbers involved in interval arithmetic.
- Implementation:
  - our own Ocaml implementation using integers,
  - with MPFR about threefold speedup.

# Dyadic intervals

The poset of dyadic intervals $\mathbb{ID}$:

- A dyadic interval $[c - r, c + r]$ is represented by its center $c \in \mathbb{D}$ and radius $r \in \mathbb{D}$.
- No need to have very precise $c$ when $r$ is large.
- No need to have very precise $r$.
- So we always *normalize* intervals to have $r$ with small numerator (e.g. 32 bits) and suitably rounded $c$. This trades a little bit of precision for quite a bit of space and time.

## $\omega$-cpos

- We formalize $\omega$-cpos generated from a base.
- Take care to get continuous domains, not algebraic ones, and consider only chains, rather than general directed sets.
- *Crucial:* how do we say that $P \subseteq D$ is a base for domain $D$?
  - Inefficient: every $x \in D$ is the supremum of a chain in $P$.
  - Broken: every $x \in D$ is the supremum of a sequence in $P$.
  - Right: for every $x \in D$ there exists $(a_k)_k \in P$ such that

$$\forall k \in \mathbb{N}.\, a_k \leq x \quad \text{and} \quad \lim_k a_k = x \text{ (in Scott topology)}$$

(For $D = \mathbb{IR}$ and $P = \mathbb{ID}$ this condition was given by Lambov, without using domains explicitly.)

# Reals as maximal elements of the interval domain

- The interval domain $\mathbb{IR}$ is the completion of $\mathbb{ID}$.
- The reals are a Cauchy-complete, Archimedean ordered field.
- $\mathbb{IR}$ is a domain model for $\mathbb{R}$.
- Avoiding explict rates of convergence:
  - RZ translates the Archimedean axiom $\forall x \in \mathbb{R}. \forall k \in \mathbb{N}. \exists d \in \mathbb{D}. |x - d| < 2^{-k}$ to specification for

    $$\texttt{approx} : \texttt{real} \to \texttt{int} \to \texttt{dyadic}$$

    computing $d$ from $x$ and $k$ such that $|x - d| < 2^{-k}$.
  - The axiom stating that $\mathbb{IR}$ is generated by $\mathbb{ID}$ yields

    $$\texttt{stage} : \texttt{real} \to \texttt{int} \to \texttt{dyadicInterval}$$

    such that `stage` $x$ is a sequence of intervals converging to $x$, without a prescribed speed of convergence.

# In conclusion

- Other issues not discussed:
  - Strict linear order $<$ on $\mathbb{R}$ as a map into the $\omega$-cpo of partial booleans.
  - How do we represent continuous real maps?
- Statistics:
  - 503 lines of formal theories in RZ.
  - 1022 lines of Ocaml implementation.
  - 10 times slower than iRRAM for basic arithmetic.
- Lessons learned:
  - Domain theory *does* play a role in state-of-the-art exact real arithmetic.
  - Theoreticians' sense of elegance *can* harm efficiency.
  - Don't compete with programmers—teach them new ideas!

# Future directions

- Implement something new—it will be horribly inefficient:
  - the next big step is to implement locally *non*-compact spaces, e.g., Banach spaces $\mathcal{C}^{(k)}(\mathbb{R})$, $L^p$, $\ell^p$, …
- Invent new ways of computing with real numbers, higher types and hyperspaces.
- A promising direction is "computation with quantifiers":
  - Paul Taylor's Abstract Stone Duality (ASD) and use of $\exists x \in \mathbb{R}$ and $\forall x : [a, b]$ in real analysis.
  - The Russian version of ASD seems to be $\Sigma$-definability (with $\forall_K$), but ASD has more of a programming flavor.
  - Martin Escardó's implementation of $\forall_K$ in Haskell is surprisingly efficient.