Metric Spaces in Synthetic Topology

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Overview

- **Theme:**

  Explore the connections between synthetic topology and topology induced by metric.

- **Purpose:**

  Find sufficient conditions for the two topologies to match when metric is nice enough (read: complete and separable).

- **Result:**

  Sufficient (in our setting) to assume this for $\mathbb{N}^{\mathbb{N}}$.

The synthetic setting

- Work in a topos, assume number-number choice $\mathrm{AC}_{0,0}$.
- Every set (object) X is naturally equipped with an intrinsic topology Σ^X.
- Σ is a dominance (more details later).
- **Reminder:** In such a setting, all maps are continuous.

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- Consider a set X with a metric $d : X \times X \to \mathbb{R}$.
- Say that X is metrized by d when the topology induced by d matches the intrinsic topology Σ^X.
- But what should "induced topology" mean synthetically?

Topology induced by metric

- Consider a space X and a metric $d : X \times X \to \mathbb{R}$.
- A (metric) ball is $B(x, r) = \{y \in X \mid d(x, y) < r\}$.
- **Classically:**

  $U \subseteq X$ is open in the topology induced by d if and only if U is a union of metric balls.

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- Under mild assumptions, balls are intrinsically open, consequently overt unions of balls are open.
- **Synthetically**, define:

  $U \subseteq X$ is metric open if and only if it is an overt union of metric balls.

- **Definition:** (The topology of) a set X is metrized by d when metric open sets coincide with open sets.

Connection between intrinsic and metric topology

- Typically, intrinsic topology is finer than metric topology.
- **Example 1:**

  If $\Sigma = \Omega$ all subsets of $\mathbb{R}$ are open, but many are not metric open for the Euclidean metric.

- We might blame Example 1 on unreasonable choice of Σ, however:

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- **Example 2:**

  In the effective topos, the usual Σ is very nice but there still exist open subsets of $\mathbb{R}$ which are not metric open.

- Thus, rather than imposing conditions directly on Σ, we take a different approach.

Transfer by metric open maps

- **Definition:** A map $f : (X, d_X) \to (Y, d_Y)$ is metric open when it maps metric open subsets of X to metric open subsets of Y.
- **Proposition:**

  If X is metrized by d_X and there exists a metric open surjection $f : (X, d_X) \to (Y, d_Y)$, then Y is metrized by d_Y.

- **Proof:** Take $U \subseteq Y$ open. As f is continuous, $f^{-1}(U)$ is open, hence metric open in X. Since f is a metric open surjection, $U = f(f^{-1}(U))$ is metric open in Y.

Σ, $\mathbb{N}$, $\mathbb{R}$, $\mathbb{Q}$

- In addition to $AC_{0,0}$ we require:
  - Σ is a dominance with $\bot, \top \in \Sigma$,
  - $\mathbb{N}$ is overt.
- Observe:
  - Σ is a lattice with countable $\bigvee$ that distribute over finite $\wedge$.
  - Since $\mathbb{N}$ has decidable equality, it is discrete and Hausdorff.
  - Dedekind and Cauchy reals coincide because of $AC_{0,0}$.
  - Relation $<$ is open in $\mathbb{R} \times \mathbb{R}$ because $\mathbb{N}$ is overt.

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- $\mathbb{Q}$ is a decidable field. At least two metrics on $\mathbb{Q}$:

$$d_E(r, s) = |r - s| \qquad \text{(Euclidean)}$$

$$d_D(r, s) = (\text{if } r = s \text{ then } 0 \text{ else } 1) \qquad \text{(discrete)}$$

Which is "better"?

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  Which is "better"?
- Topology induced by d_E is strictly weaker than $\Sigma^{\mathbb{Q}}$.
- Topology induced by d_D is $\Sigma^{\mathbb{Q}}$. We prefer this one.

Complete separable metric spaces (CSM)

- Baire space $\mathbb{N}^{\mathbb{N}}$ with comparison metric

$$d_C(\alpha, \beta) = 2^{-\min_k(\alpha_k \neq \beta_k)}$$

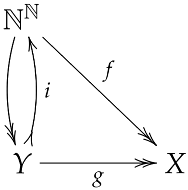
  is the prototypical CSM.
- **Spread Representation Theorem:**

  Every CSM is a metric continuous image of $\mathbb{N}^{\mathbb{N}}$*.*

  (The proof uses $\mathrm{AC}_{0,0}$.)
- Can we use the theorem to transfer metrizability of $\mathbb{N}^{\mathbb{N}}$ to other CSMs?

Construction of surjective $f : \mathbb{N}^{\mathbb{N}} \to X$

- Unfortunately, the surjection $f : \mathbb{N}^{\mathbb{N}} \to X$ given by the Spread Representation Theorem need not be metric open.
- The map f is constructed as $f = g \circ r$



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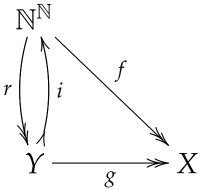

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- The map g turns out to be a metric open surjection!
- But r need not be metric open …
- To overcome this, we use the fact that $(\mathbb{N}^{\mathbb{N}}, d_C)$ is ultrametric.

Retracts of ultrametric spaces

Proposition:

If Z is overt and metrized by an ultrametric then every retract of Z is metrized in the induced metric.

Note: In an ultrametric space every point in a ball is its centre.

Proof: Given $Y \subseteq Z$ with a retraction $r : Z \to Y$, consider $U \in \Sigma^Y$. Then $r^{-1}(U) = \bigcup_{i \in I} B_Z(x_i, \epsilon_i)$ with I overt. The set

$$K = \{(i, y) \in I \times Y \mid y \in B_Z(x_i, \epsilon_i)\}$$

is overt and so

$$U = Y \cap r^{-1}(U) = \bigcup_{i \in I} Y \cap B_Z(x_i, \epsilon_i) = \bigcup_{(i,y) \in K} Y \cap B_Z(x_i, \epsilon_i) =$$
$$= \bigcup_{(i,y) \in K} Y \cap B_Z(y, \epsilon_i) = \bigcup_{(i,y) \in K} B_Y(y, \epsilon_i) \ .$$

Putting all this together

Theorem:

If $\mathbb{N}^{\mathbb{N}}$ is metrized by d_C then every CSM is metrized by its metric (and the converse holds trivially).

In view of this, we suggest:

Axiom:

The topology of $\mathbb{N}^{\mathbb{N}}$ is induced by the comparison metric d_C.

First consequences of the axiom

The axiom ensures a well behaved theory of CSMs.

- Up to topological equivalence, a set has at most one complete separable metric (which then induces the intrinsic topology).
- CSMs are overt.
- **Continuity Principle:** For a CSM X and metric Y, every map $f : X \to Y$ is metric continuous.

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What can we say about more general spaces, e.g., T_0 spaces?

- There is a surjection $q : \mathbb{N}^{\mathbb{N}} \to \Sigma^{\mathbb{N}}$ mapping balls to basic opens for the Scott topology, provided one-point space is countably based, i.e., $\Sigma = \Sigma^0_1$. In this case:
- **Scott's principle:** The topology of $\Sigma^{\mathbb{N}}$ is the Scott topology.

Concluding remarks

- We used number choice. Can we avoid it?
- The axiom implies that the *Cantor space* $2^{\mathbb{N}}$ has the metric topology. Does it also imply that $2^{\mathbb{N}}$ is (synthetically) compact?