Continuity begets Continuity (A Theorem about Douglas Bridges)

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Proving sequential continuity

A recent theorem of Bishop-style constructivism:

Theorem (Bauer & Simpson 2004)

Every sequentially continuous $f : \mathbb{Z}^{\mathbb{N}} \to \mathbb{Z}$ *extends to a sequentially continuous* $h : \mathbb{R}^{\mathbb{N}} \to \mathbb{R}$*:*

$$\begin{array}{ccc} \mathbb{Z}^{\mathbb{N}} & \xrightarrow{f} & \mathbb{Z} \\ {\scriptstyle\subseteq}\downarrow & & \downarrow{\scriptstyle\subseteq} \\ \mathbb{R}^{\mathbb{N}} & \xrightarrow[h]{} & \mathbb{R} \end{array}$$

In the proof we first construct h from f, then prove that h is sequentially continuous. But if Douglas Bridges constructs h, is it not automatically sequentially continuous?

A known theorem about *a* Douglas Bridges

Theorem

There is an alternative universe containing a Douglas Bridges, in which all functions are sequentially continuous.

- Alas, this theorem is about a *different* Douglas Bridges who never speaks to, say, classical mathematicians.
- *Our* Douglas Bridges travels between universes. We want a theorem that would allow him to skip proofs of sequential continuity and focus on other, more important things in life.

A desired scenario

Location: Classical Universe.
Douglas Bridges talks to a Classical Scientist:

KW: *I want to solve a functional system of equations involving maps g_i and an unknown map f.*

DB: *What do you know about the g_i's?*

KW: *They are continuous.*

DB: (after some thought) *I can prove your system has a solution f.*

KW: *But I need a continuous f.*

DB: (responds immediately) *There is one.*

KW: *How do you know?*

DB: *I just do, it's a theorem about me. Now please turn on the TV, the game has already started.*

The main idea

- Suppose **S** is a model of constructive mathematics, e.g.:
  - BISH – pure Bishop's constructive mathematics
  - CLASS – classical mathematics
  - INT – Brouwer's intuitionism
  - RUSS – Russian recursive mathematics
  - Various realizability models
- Build another model **L**, *relative to* **S**, in which all maps are sequentially continuous.
- Reinterpret in **L** the original construction of a map between complete metric spaces in **S** to conclude that it is sequentially continuous.

Comments:

- We shall make the notion of "model" precise later.
- The whole argument will be predicative and constructive.

Outline

Motivation

Weak limit spaces

Weak limit spaces as a model of constructive mathematics

Main Theorem: Continuity begets continuity

Concluding remarks

The space $\mathbb{N}^+$

- The one-point compactification of natural numbers:

$$\mathbb{N}^+ = \{a : \mathbb{N} \to \mathsf{2} \mid \forall n \in \mathbb{N} \,.\, a_n \leq a_{n+1}\} \,.$$

- We have $\mathbb{N} \subseteq \mathbb{N}^+$ with $n \in \mathbb{N}$ represented as

$$\underbrace{0, 0, \ldots, 0}_{n}, 1, 1, 1, 1, 1, \ldots$$

  The sequence $\infty = 0, 0, 0, \ldots$ is the "point at infinity".
- The set $\mathbb{N}^+$ is a complete separable metric space with metric $d(a, b) = \mathsf{2}^{-\min_k (a_k \neq b_k)}$ inherited from Cantor space $\mathsf{2}^{\mathbb{N}}$.

Convergent sequences

- In a metric space (M, d) a convergent sequence is usually considered separately from its limit:

$$(x_n)_{n\in\mathbb{N}} \,, \qquad x \in M \,, \qquad \lim_{n\to\infty} x_n = x \,.$$

- Constructively, it makes more sense to consider a single map:

$$x_- : \mathbb{N}^+ \to M \,, \qquad \lim_{n\to\infty} x_n = x_\infty \,.$$

  This way the limit is not artificially detached from the sequence.

- If (M, d) is a complete metric space, the convergent sequences in M are in 1–1 correspondence with continuous maps $\mathbb{N}^+ \to M$.

The monoid of reindexings

- The set

$$\mathcal{R} = \{r : \mathbb{N}^+ \to \mathbb{N}^+ \mid r \text{ is continuous}\} ,$$

  is a monoid for composition of functions.
- If $x_- : \mathbb{N}^+ \to M$ is a sequence, we can think of $x \circ r$ as a *reindexing* of x.
- This defines a right action of $\mathcal{R}$ on the set of convergent sequences in a complete metric space M.

Continuity and sequential continuity

- A map $f : L \to M$ between metric spaces is *sequentially continuous* when it preserves limits of convergent sequences:
$$f(\lim_{n\to\infty} x_n) = \lim_{n\to\infty} f(x_n) \ .$$
- Sequential continuity is generally weaker than the ordinary $\epsilon\delta$ continuity.
- Classically both notions agree on metric spaces.
- We always use sequential continuity.

Weak limit spaces

Definition (cf. Matthias Schröder)

A *weak limit space* is a set X with a collection $\mathcal{C}(X) \subseteq \mathbb{N}^+ \to X$ of *convergent* sequences. We write $x_n \to x_\infty$ when $x_- \in \mathcal{C}(X)$. The convergent sequences satisfy:

1. Constant sequences are convergent: $x \to x$.
2. If the tail converges, so does the sequence:
$$x_{n+1} \to x_\infty \implies x_n \to x_\infty \ .$$
3. A reindexing of a convergent sequence is convergent:
$$x_n \to x_\infty \wedge r \in \mathcal{R} \implies x_{r(n)} \to x_{r(\infty)} \ .$$

A *weak limit map* is a map $f : X \to Y$ such that $x_n \to x_\infty$ implies $f(x_n) \to f(x_\infty)$.

Complete metric spaces as weak limit spaces

- A complete metric space (M, d) is a weak limit space if we define

$$\mathcal{C}(M) = \{x_- : \mathbb{N}^+ \to M \mid x_- \text{ is continuous}\} .$$

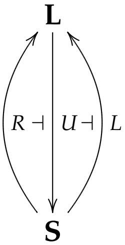
- A map $f : L \to M$ between complete metric spaces is sequentially continuous if, and only if, it is a weak limit space map.
- This defines a full and faithful embedding **CM** $\to$ **L** between categories
  - **CM**: complete metric spaces and sequentially continuous maps,
  - **L**: weak limit spaces and weak limit maps.

Sets as weak limit spaces

Thee are two ways of making a set $A \in \mathbf{S}$ into a weak limit space:

- $L(A) = \{x_- : \mathbb{N}^+ \to A \mid x_- \text{ eventually constant}\}$,
- $R(A) = \mathbb{N}^+ \to A$, the "intrinsically" convergent sequences.

These are left and right adjoints of the forgetful functor:



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- Weak limit spaces
- Weak limit spaces as a model of constructive mathematics
- Main Theorem: Continuity begets continuity
- Concluding remarks

Models of predicative constructive mathematics

A model of predicative constructive mathematics should support at least:

- Set operations: dependent products and sums, subsets, finite disjoint sums, possibly certain quotients.
- Intuitionistic first-order logic.
- Natural numbers, possibly other inductive types.
- Number Choice, or even better Dependent Choice.
- Real numbers: Cauchy-complete archimedean ordered field.

We might call such a model BRID.

Theorem

If $\mathbf{S}$ *is a model in the above sense, then so is* $\mathbf{L}$.

Exponentials in $\mathbf{L}$

For $X, Y \in \mathbf{L}$, the exponential is formed as

$$Y^X = \{f : X \to Y \mid f \text{ is a weak limit map}\} \ .$$

with convergent sequences of functions characterized by

$$f_n \to f_\infty \iff \forall\, x_n \to_X x \,.\, \forall\, r \in \mathcal{R} \,.\, f_{r(n)}(x_n) \to_Y f_{r(\infty)}(x_\infty) \ .$$

Equivalently $f_n \to f_\infty$ when the transpose $\tilde{f} : \mathbb{N}^+ \times X \to Y$ is a weak limit map. This is stronger than the classical condition

$$x_n \to_X x \implies f_n(x_n) \to_Y f_\infty(x_\infty)$$

in which reindexing is omitted.

First-order intuitionistic logic in **L**

Informally speaking, a proposition is valid in **L** if it is valid in **S** point-wise and "convergent-sequence-wise", e.g., given $e : X \to Y$ in **L**, the proposition

$$\forall y \in Y \,.\, \exists x \in X \,.\, e(x) = y$$

is valid in **L** if in **S**

- for every $y \in Y$ there is $x \in X$ such that $e(x) = y$, and
- for every $y_n \to_Y y_\infty$ there is $x_n \to_X x_\infty$ such that $e(x_n) = y_n$ for all $n \in \mathbb{N}^+$.

Outline

Complete metric spaces in **L**

For our purposes, a crucial property of **L** is:

Theorem (Transfer of C(S)M's)

Complete (separable) metric spaces in **S** *are the same as complete (separable) metric spaces in* **L**.

Recall the embedding $I : \mathbf{CM} \to \mathbf{L}$. The proof of the transfer of C(S)M's involves checking that:

- $I(\mathbb{N})$ are the natural numbers in **L**.
- $I(\mathbb{R})$ is a Cauchy-complete archimedean ordered field in **L**.
- If M is complete (separable) metric space in **S** then $I(M)$ is of the same kind in **L**, and vice versa.

The Main Theorem

Theorem (Continuity begets continuity)

Suppose g_i are sequentially continuous maps between complete (separable) metric spaces in **S***, and $\Phi(g_1, \ldots, g_n, f)$ is a functional system of equations. If BRID proves that*

$$\exists f : L \to M \,.\, \Phi(g_1, \ldots, g_n, f) \tag{1}$$

where L and M are complete metric spaces, then there exists a sequentially continuous $h : L \to M$ such that $\Phi(g_1, \ldots, g_n, h)$.

Proof.

Reinterpret the BRID proof in **L** to conclude that (1) is valid in **L**. From this the existence of desired sequentially continuous h in **S** follows. □

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- Can we find a better and more general formulation of the Main Theorem?
- As a present to Douglas, we have swept category theory under the rug.
- We expect to be able to treat pointwise continuity by switching to the monoid of continuous maps on Baire space.
- With powersets thrown in, a topos-theoretic sheaf construction accomplishes an analogous result (cf. Johstone's topological topos).