First Steps in Synthetic Computability

Andrej Bauer

Department of Mathematics and Physics
University of Ljubljana
Slovenia

MATHLOGAPS, Fischbachau, September 2005

How cool is computability theory?

- Way cool:
  - surprising theorems
  - clever programs
  - clever proofs
- Way horrible, it contains expressions like

$$\varphi_{p(r(i,\varphi_{q(i)}(\hat{g}(n,i,m)+1),m),\varphi_{q(i)}(\hat{g}(n,i,m)-1))}(a - \hat{g}(n,i,m))$$

- Can we do computability theory as “ordinary” math?
  - use axiomatic method
  - argue conceptually and abstractly
  - use customary mathematical notions

Related Work

- Friedman [1971], axiomatizes coding and universal functions
- Moschovakis [1971] & Fenstad [1974], axiomatize computations and subcomputations
- Hyland [1982], effective topos
- Richman [1984], an axiom for effective enumerability of partial functions
- We shall follow Richman [1984] in style, and borrow ideas from Rosolini [1986], Berger [1983], and Spreen [1998].

Computability without Turing Machines

- Use ordinary set theory:
  no Turing Machines, or other special notions.
- Add a couple of axioms about sets of numbers.
- The underlying logic is *intuitionistic*:
  this is a theorem, not a political conviction.
- Interpretation in the effective topos translates our theory back to classical recursion theory.

Basic setup

- Intuitionistic logic:
  generally, no Law of Excluded Middle or Proof by Contradiction.
- As in Bishop-style constructive mathematics, we do not accept the full Axiom of Choice, but only Number Choice (and Dependent Choice).
- Basic sets:

$$\emptyset, \quad \mathsf{1} = \{*\}, \quad \mathbb{N} = \{0, 1, 2, \ldots\}$$

- Set operations:

$$A \times B, \quad A + B, \quad B^A = A \to B, \quad \{x \in A \mid p(x)\}, \quad \mathcal{P}A$$

- We say that A is
  - *non-empty* if $\neg\forall\, x \in A\,.\,\bot$,
  - *inhabited* if $\exists\, x \in A\,.\,\top$.

Some interesting sets

- The set of truth values:

$$\Omega = \mathcal{P}\mathsf{1}$$
$$\text{truth } \top = \mathsf{1}, \quad \text{falsehood } \bot = \emptyset$$

- The set of *decidable* truth values:

$$\mathsf{2} = \{0, 1\} = \{p \in \Omega \mid p \vee \neg p\} ,$$

  where we write $1 = \top$ and $0 = \bot$.
- The set of *classical* truth values:

$$\Omega_{\neg\neg} = \{p \in \Omega \mid \neg\neg p = p\} .$$

- $\mathsf{2} \subseteq \Omega_{\neg\neg} \subseteq \Omega$.

Decidable and classical sets

- A subset $S \subseteq A$ is equivalently given by its characteristic map $\chi_S : A \to \Omega$, $\chi_S(x) = (x \in S)$.
- A subset $S \subseteq A$ is *decidable* if $\chi_S : A \to 2$, equivalently

$$\forall x \in A \,.\, (x \in S \vee x \notin S) \enspace .$$

- A subset $S \subseteq A$ is *classical* if $\chi_S : A \to \Omega_{\neg\neg}$, equivalently

$$\forall x \in A \,.\, (\neg(x \notin S) \implies x \in S) \enspace .$$

The generic convergent sequence

- A useful set is the *generic convergent sequence*:

$$\mathbb{N}^+ = \{a \in 2^{\mathbb{N}} \mid \forall k \in \mathbb{N} \,.\, a_k \leq a_{k+1}\} \,.$$

- We have $\mathbb{N} \subseteq \mathbb{N}^+$ via $n \mapsto \lambda k. \, (k \leq n)$.
- But there is also $\infty = \lambda k. \, 0$.
- $\mathbb{N}^+$ can be thought of as the one-point compactification of $\mathbb{N}$.

Enumerable & finite sets

- A is *finite* if there exist $n \in \mathbb{N}$ and an onto map $e : \{1, \ldots, n\} \twoheadrightarrow A$, called a *listing* of A. An element may be listed more than once.
- A is *enumerable (countable)* if there exists an onto map $e : \mathbb{N} \twoheadrightarrow \mathtt{1} + A$, called an *enumeration* of A. For inhabited A we may take $e : \mathbb{N} \twoheadrightarrow A$.
- A is *infinite* if there exists an injective $a : \mathbb{N} \rightarrowtail A$.

Lawvere $\to$ Cantor

Theorem (Lawvere)

If $e : A \to B^A$ is onto then B has the fixed point property.

Proof.

Given $f : B \to B$, there is $x \in A$ such that
$e(x) = \lambda y : A \,.\, f(e(y)(y))$. Then $e(x)(x) = f(e(x)(x))$. □

Corollary (Cantor)

There is no onto map $e : A \twoheadrightarrow \mathcal{P}A$.

Proof.

$\mathcal{P}A = \Omega^A$ and $\neg : \Omega \to \Omega$ does not have a fixed point. □

Non-enumerability of Cantor and Baire space

Corollary

$2^{\mathbb{N}}$ *and* $\mathbb{N}^{\mathbb{N}}$ *are not enumerable.*

Proof.

2 and $\mathbb{N}$ do not have the fixed-point property. □

We have proved our first synthetic theorem: there are no effective enumerations of recursive sets and total recursive functions.

Projection Theorem

Recall: the *projection* of $S \subseteq A \times B$ is the set

$$\{x \in A \mid \exists y \in B \,.\, \langle x, y \rangle \in S\} \ .$$

Theorem (Projection)

A subset of $\mathbb{N}$ is enumerable iff it is the projection of a decidable subset of $\mathbb{N} \times \mathbb{N}$.

Proof.

If A is enumerated by $e : \mathbb{N} \to \mathtt{1} + A$ then A is the projection of the *graph* of e.
If A is the projection of $B \subseteq \mathbb{N} \times \mathbb{N}$, define $e : \mathbb{N} \times \mathbb{N} \to \mathtt{1} + A$ by

$$e\langle m, n\rangle = \mathtt{if}\ \langle m, n\rangle \in B\ \mathtt{then}\ m\ \mathtt{else}\ \star \ . \qquad \square$$

Semidecidable sets

- A *semidecidable truth value* $p \in \Omega$ is one of the form, for some $d : \mathbb{N} \to \mathsf{2}$,
$$p = \exists\, n \in \mathbb{N} \,.\, d(n)\ .$$
- The set of semidecidable truth values:
$$\Sigma = \{p \in \Omega \mid \exists\, d \in \mathsf{2}^{\mathbb{N}} \,.\, p = \exists\, n \in \mathbb{N} \,.\, d(n)\}\ .$$

  This is Rosolini's *dominance*.
- $\mathsf{2} \subseteq \Sigma \subseteq \Omega$.
- A subset $S \subseteq \mathbb{N}$ is *semidecidable* if $\chi_S : A \to \Sigma$.

Σ as a quotient of $\mathbb{N}^+$

- Σ is a quotient of $2^{\mathbb{N}}$ via taking countable joins: $d \in 2^{\mathbb{N}}$ is mapped to $\exists\, n \in \mathbb{N} \,.\, d(n)$.
- Σ is a quotient of $\mathbb{N}^+$ via the map $q : \mathbb{N}^+ \to \Sigma$, defined by $q(t) = (t < \infty)$.
- If $q(t) = s$ we say that t is a *time at which s becomes true*. Beware, t is not uniquely determined!

Semidecidable subsets

Theorem

The enumerable subsets of $\mathbb{N}$ are precisely the semidecidable subsets of $\mathbb{N}$.

Proof.

By Projection Theorem, an enumerable $A \subseteq \mathbb{N}$ is the projection of a decidable $B \subseteq \mathbb{N} \times \mathbb{N}$. Then $n \in A$ iff $\exists\, m \in \mathbb{N}\,.\, \langle n, m \rangle \in B$. Conversely, if $A \in \Sigma^{\mathbb{N}}$, by Number Choice there is $d : \mathbb{N} \times \mathbb{N} \to 2$ such that $n \in A$ iff $\exists\, m \in \mathbb{N}\,.\, d(m, n)$. $\square$

The enumerable subsets of $\mathbb{N}$:

$$\mathcal{E} = \Sigma^{\mathbb{N}}\ .$$

The Topological View

- Σ is the *Sierpinski space*.
- Σ is closed under finite meets, enumerable joins, and finite meets distribute over enumerable joins.
- A *σ-frame* is a lattice with enumerable joins that distribute over finite meets.
- The *topology of A* is Σ^A.

Partial functions

- A partial function $f : A \rightharpoonup B$ is a function $f : A' \to B$ defined on a subset $A' \subseteq A$, called the *domain* of f.
- Equivalently, it is a function $f : A \to \widetilde{B}$, where

$$\widetilde{B} = \{s \in \mathcal{P}B \mid \forall x, y \in B \,.\, (x \in s \land y \in s \implies x = y)\} \,.$$

- The singleton map $\{-\} : B \to \widetilde{B}$ embeds B in $\widetilde{B}$.
- For $s \in \widetilde{B}$, write $s\downarrow$ when s is inhabited.
- Which partial functions $\mathbb{N} \to \widetilde{\mathbb{N}}$ have enumerable graphs?

Σ-partial functions

Proposition

$f : \mathbb{N} \to \widetilde{\mathbb{N}}$ has an enumerable graph iff $f(n){\downarrow} \in \Sigma$ for all $n \in \mathbb{N}$.

Define the *lifting* operation

$$A_\perp = \{s \in \widetilde{A} \mid s{\downarrow} \in \Sigma\} .$$

For $f : A \to B$ define $f_\perp : A_\perp \to B_\perp$ to be

$$f_\perp(s) = \{f(x) \mid x \in s\} .$$

A Σ-*partial function* is a function $f : A \to B_\perp$.

Domains of Σ-partial functions

Proposition

A subset is semidecidable iff it is the domain of a Σ-partial function.

Proof.

A semidecidable subset $S \in \Sigma^A$ is the domain of its characteristic map $\chi_S : A \to \Sigma = 1_\perp$.

If $f : A \to B_\perp$ is Σ-partial then its domain is the set $\{x \in A \mid f(x)\downarrow\}$, which is obviously semidecidable. □

The Single-Value Theorem

A *selection* for $R \subseteq A \times B$ is a partial map $f : A \rightharpoonup B$ such that, for every $x \in A$,

$$\exists y \in B \,.\, R(x, y) \implies f(x){\downarrow} \wedge R(x, f(x)) \,.$$

This is like a choice function, expect it only chooses when there is something to choose from.

Theorem (Single Value)

Every open relation $R \in \Sigma^{\mathbb{N}\times\mathbb{N}}$ has a Σ-partial selection.

Axiom of Enumerability

Axiom (Enumerability)

There are enumerably many enumerable sets of numbers.

Let $\mathsf{W} : \mathbb{N} \twoheadrightarrow \mathcal{E}$ be an enumeration.

Proposition

Σ *and* $\mathcal{E}$ *have the fixed-point property.*

Proof.

By Lawvere, $\mathsf{W} : \mathbb{N} \twoheadrightarrow \mathcal{E} = \Sigma^{\mathbb{N}} \cong \Sigma^{\mathbb{N}\times\mathbb{N}} \cong \mathcal{E}^{\mathbb{N}}$. □

The Law of Excluded Middle Fails

The Law of Excluded Middle says $\mathsf{2} = \Omega$.

Corollary

The Law of Excluded Middle is false.

Proof.

Among the sets $\mathsf{2} \subseteq \Sigma \subseteq \Omega$ only the middle one has the fixed-point property, so $\mathsf{2} \neq \Sigma \neq \Omega$. □

Enumerability of $\mathbb{N} \to \mathbb{N}_\perp$

Proposition

$\mathbb{N} \to \mathbb{N}_\perp$ is enumerable.

Proof.

Let $V : \mathbb{N} \twoheadrightarrow \Sigma^{\mathbb{N}\times\mathbb{N}}$ be an enumeration. By Single-Value Theorem and Number Choice, there is $\varphi : \mathbb{N} \to (\mathbb{N} \to \mathbb{N}_\perp)$ such that φ_n is a selection of V_n. The map φ is onto, as every $f : \mathbb{N} \to \mathbb{N}_\perp$ is the only selection of its graph. □

Corollary (Church's Thesis)

$\mathbb{N}^\mathbb{N}$ is subcountable (because $\mathbb{N}^\mathbb{N} \subseteq \mathbb{N}_\perp^\mathbb{N}$).

In other words, $\forall f \in \mathbb{N}^\mathbb{N} \,.\, \exists n \in \mathbb{N} \,.\, f = \varphi_n$.

Focal sets

- A *focal set* is a set A together with a map $\epsilon_A : A_\perp \to A$ such that $\epsilon_A(\{x\}) = x$ for all $x \in A$:

$$\begin{array}{ccc} A & \xrightarrow{\{-\}} & A_\perp \\ & \diagdown\!\!\diagdown & \downarrow{\scriptstyle \epsilon_A} \\ & & A \end{array}$$

  The *focus* of A is $\perp_A = \epsilon_A(\perp)$.

- A lifted set $A_\perp$ is always focal (because lifting is a monad whose unit is $\{-\}$).

Enumerable focal sets

- Enumerable focal sets, known as *Eršov complete sets*, have good properties.
- A *flat domain* $A_\perp$ is focal. It is enumerable if A is decidable and enumerable.
- If A is enumerable and focal then so is $A^{\mathbb{N}}$:

$$\mathbb{N} \xrightarrow{\varphi} \mathbb{N}_\perp^{\mathbb{N}} \xrightarrow{e_\perp^{\mathbb{N}}} A_\perp^{\mathbb{N}} \xrightarrow{\epsilon_A^{\mathbb{N}}} A^{\mathbb{N}}$$

- Some enumerable focal sets are

$$\Sigma^{\mathbb{N}}, \quad 2_\perp^{\mathbb{N}}, \quad \mathbb{N}_\perp^{\mathbb{N}} .$$

Topological Exterior and Creative Sets

- The *exterior* of an open set is the largest open set disjoint from it.
- An open set $U \in \Sigma^A$ is *creative* if it is without exterior: every $V \in \Sigma^A$ disjoint from U can be enlarged and still be disjoint from U.

Theorem

There exists a creative subset of $\mathbb{N}$.

Proof.

The familiar $K = \{n \in \mathbb{N} \mid n \in \mathsf{W}_n\}$ is creative. Given any $V \in \mathcal{E}$ with $V = \mathsf{W}_k$ and $K \cap V = \emptyset$, we have $k \notin V$, so $V' = V \cup \{k\}$ is larger and still disjoint from K. □

Immune and Simple Sets

- A set is *immune* if it is neither finite nor infinite.
- A set is *simple* if it is open and its complement is immune.

Theorem

There exists a closed subset of $\mathbb{N}$ *which is neither finite nor infinite.*

Proof.

Following Post, consider $P = \{\langle m, n\rangle \in \mathbb{N} \times \mathbb{N} \mid n > 2m \wedge n \in W_m\}$, and let $f : \mathbb{N} \to \mathbb{N}_\perp$ be a selection for P. Then $S = \{n \in \mathbb{N} \mid \exists m \in \mathbb{N} \,.\, f(m) = n\}$ is the complement of the set we are looking for.
Because $f(m) > 2m$ the set $\mathbb{N} \setminus S$ cannot be finite.
For any infinite enumerable set $U \subseteq \mathbb{N} \setminus S$ with $U = \mathsf{W}_m$, we have $f(m)\downarrow$, $f(m) \in \mathsf{W}_m = U$, and $f(m) \in S$, hence U is not contained in $\mathbb{N} \setminus S$. □

Inseparable sets

Theorem

There exists an element of Plotkin's $\mathsf{2}_\perp^{\mathbb{N}}$ that is inconsistent with every maximal element of $\mathsf{2}_\perp^{\mathbb{N}}$.

Proof.

Because $\mathsf{2}_\perp$ is focal and enumerable, $\mathsf{2}_\perp^{\mathbb{N}}$ is as well. Let $\psi : \mathbb{N} \twoheadrightarrow \mathsf{2}_\perp^{\mathbb{N}}$ be an enumeration, and let $t : \mathsf{2}_\perp \to \mathsf{2}_\perp$ be the isomorphism $t(x) = \neg_\perp x$ which exchanges 0 and 1, and fixes $\perp$. Consider $a \in \mathsf{2}_\perp^{\mathbb{N}}$ defined by $a(n) = t(\psi_n(n))$. If $b \in \mathsf{2}_\perp^{\mathbb{N}}$ is maximal with $b = \psi_k$, then $a(k) = \neg\psi_k(k) = \neg b(k)$. Because $a(k)$ and $b(k)$ are both total and different they are inconsistent. Hence a and b are inconsistent. □

End of Part I

Let's get some coffee.

Part II

1. Quick overview of Part I
2. Post's Theorem and Markov Principle
3. Recursion Theorem
4. Rice-Shapiro & Myhill-Shepherdson
5. Recursive Analysis

Recall from Part I

Truth values:

- truth values $\Omega = \mathcal{P}1$,
- decidable truth values $2 = \{p \in \Omega \mid p \lor \neg p\}$,
- classical truth values $\Omega_{\neg\neg} = \{p \in \Omega \mid \neg\neg p = p\}$,
- semidecidable truth values

$$\Sigma = \{p \in \Omega \mid \exists d \in 2^{\mathbb{N}} \,.\, p = (\exists n \in \mathbb{N} \,.\, d(n) = 1)\} \ .$$

Enumerable, or semidecidable, subsets of $\mathbb{N}$:

$$\mathcal{E} = \Sigma^{\mathbb{N}} \ .$$

Σ-partial functions: $\mathbb{N} \to \mathbb{N}_{\perp}$.

Axiom of Enumerability

Axiom (Enumerability)

There are enumerably many enumerable sets of numbers.

An enumeration $\mathsf{W} : \mathbb{N} \twoheadrightarrow \mathcal{E}$.
Consequences:

- Σ and $\mathcal{E}$ have the fixed-point property,
- Law of Excluded Middle is false,
- $\mathbb{N} \to \mathbb{N}_\perp$ is enumerable,
- Other enumerable sets:
  - A focal and enumerable $\implies$ $A^\mathbb{N}$ focal and enumerable,
  - $\mathbb{N} \to \mathsf{2}_\perp$ is enumerable,
  - retract of an enumerable set is enumerable,
  - Scott domains are enumerable,
- Creative, simple, immune and inseparable sets exist.

Markov Principle

- If a binary sequence $a \in 2^{\mathbb{N}}$ is not constantly 0, does it contain a 1?
- For $p \in \Sigma$, does $p \neq \bot$ imply $p = \top$?
- Is $\Sigma \subseteq \Omega_{\neg\neg}$?
- For $x \in \mathbb{N}^+$, if $x \neq \infty$ is $x = k$ for some $k \in \mathbb{N}$?

Axiom (Markov Principle)

A binary sequence which is not constantly 0 *contains a* 1.

Post's Theorem

Theorem

For all $p \in \Omega$,

$$p \in \mathsf{2} \iff p \in \Sigma \wedge \neg p \in \Sigma \,.$$

Proof.

- $\Rightarrow$ If $p \in \mathsf{2}$ then $\neg p \in \mathsf{2}$, therefore $p, \neg p \in \mathsf{2} \subseteq \Sigma$.
- $\Leftarrow$ If $p \in \Sigma$ and $\neg p \in \Sigma$ then $p \vee \neg p \in \Sigma \subseteq \Omega_{\neg\neg}$, therefore

$$p \vee \neg p = \neg\neg(p \vee \neg p) = \neg(\neg p \wedge \neg\neg p) = \neg\bot = \top \,,$$

as required.

□

Multi-valued functions

- A *multi-valued function* $f : A \rightrightarrows B$ is a function $f : A \to \mathcal{P}B$ such that $f(x)$ is inhabited for all $x \in A$.
- This is equivalent to having a *total relation* $R \subseteq A \times B$. The connection between f and R is

$$f(x) = \{y \in B \mid R(x, y)\}$$
$$R(x, y) \iff y \in f(x)\ .$$

- A *fixed point* of $f : A \rightrightarrows A$ is $x \in A$ such that $x \in f(x)$.

Recursion Theorem

Theorem (Recursion Theorem)

Every $f : A \rightrightarrows A$ on enumerable focal A has a fixed point.

Corollary (Classical Recursion Theorem)

For every $f : \mathbb{N} \to \mathbb{N}$ there is $n \in \mathbb{N}$ such that $\varphi_{f(n)} = \varphi_n$.

Proof.

In Recursion Theorem, take the enumerable focal set $A = \mathbb{N}_\perp^\mathbb{N}$ and the multi-valued function

$$F(g) = \{h \in \mathbb{N}_\perp^\mathbb{N} \mid \exists\, n \in \mathbb{N} \,.\, g = \varphi_n \wedge h = \varphi_{f(n)}\}\ .$$

There is g such that $g \in F(g)$. Thus there exists $n \in \mathbb{N}$ such that $\varphi_n = g = h = \varphi_{f(n)}$. □

Open subsets of $\mathbb{N}^+$

Lemma

If $U \in \Sigma^{\mathbb{N}^+}$ and $\infty \in U$ then there is $n \in \mathbb{N}$ such that $n \in U$.

Proof.

Suppose $\infty \in U \in \Sigma^{\mathbb{N}^+}$. By Markov Principle, it suffices to show $\neg \forall n \in \mathbb{N} \,.\, n \notin U$. So suppose $\forall n \in \mathbb{N} \,.\, n \notin U$. Recall the quotient map $q : \mathbb{N}^+ \twoheadrightarrow \Sigma$, $q : x \mapsto (x < \infty)$, and define $f : \Sigma \to \Sigma$ by $f(q(x)) = U(x)$. Now $f(\top) = \bot$ and $f(\bot) = \top$. Since Σ has the fixed-point property, there exists $p \in \Sigma$ such that $f(p) = p$. But then $p \neq \top$ and $p \neq \bot$, i.e., $\neg p \wedge \neg\neg p$, a contradiction. □

Note: the conclusion of the lemma *cannot* be improved to $\exists\, n \in \mathbb{N} \,.\, [n, \infty] \subseteq U$.

ω-Chain Complete Posets

- An *ω-chain complete poset (ω-cpo)* is a poset in which enumerable ascending chains have suprema.
- A *base* for an ω-cpo $(A, \leq)$ is an enumerable subset $S \subseteq A$ such that:
  - For all $x \in S, y \in A, (x \leq y) \in \Sigma$.
  - Every $x \in A$ is the supremum of a chain in S.
- Examples of ω-cpos:
  $\Sigma^{\mathbb{N}}, \mathbb{N} \to \mathbb{N}_\perp, \mathbb{N} \to 2_\perp$, Scott domains, ...

The Topology of ω-cpos

Theorem

1. *The open subsets of an ω-cpo are upward closed and inaccessible by chains.*
2. *If an ω-cpo A has a base S, then every open is a union of basic opens sets $\uparrow x = \{y \in A \mid x \leq y\}$, $x \in S$.*

Proof.

We only prove "upward closed": if $x \leq y$ and $x \in U \in \Sigma^A$, define $a : \mathbb{N}^+ \to A$ by

$$a_p = \bigvee_{k \in \mathbb{N}} \texttt{if } k < p \texttt{ then } x \texttt{ else } y$$

Then $a_\infty = x \in U$ and by Lemma there is $k \in \mathbb{N}$ such that $y = a_k \in U$, too. $\square$

Binary Trees

- Let 2^* be the set of finite binary sequences, with prefix-ordering $\preceq$.
- The *length* of $[a_0, \ldots, a_{n-1}] \in 2^*$ is $|a| = n$.
- A *tree* $T \subseteq 2^*$ is an inhabited prefix-closed subset.
- A *Kleene tree* T_K is a tree such that:
  1. T_K is decidable (as a subset of 2^*),
  2. T_K is unbounded: $\forall\, k \in \mathbb{N} \,.\, \exists\, a \in T_K \,.\, |a| \geq k$,
  3. every infinite path exits T_K:

$$\forall\, \alpha \in 2^{\mathbb{N}} \,.\, \exists\, n \in \mathbb{N} \,.\, [\alpha_0, \ldots, \alpha_n] \notin T_K \,.$$

Construction of a Kleene Tree

1. Recall an enumeration $\psi : \mathbb{N} \twoheadrightarrow 2^{\mathbb{N}}_{\perp}$ and $s(n) = \neg_{\perp}\psi_n(n)$ which is inconsistent with every $\alpha \in 2^{\mathbb{N}}$.
2. Let $\langle m_{-}, d_{-} \rangle : \mathbb{N} \to \mathbb{N} \times 2$ be an enumeration of the graph of s, i.e., $s(m_k) = d_k$ for all $k \in \mathbb{N}$ and we enumerate all such pairs.
3. Given $a = [a_0, \ldots, a_n] \in 2^*$, say that *$a$ clashes with $\langle m_{-}, d_{-} \rangle$*, if there is $k \leq n$ such that $m_k \leq n$ and $a_{m_k} \neq d_k$.
4. Define $K_T = \{a \in 2^* \mid a \text{ does not clash with } \langle m_{-}, d_{-} \rangle\}$.
5. K_T is a Kleene tree!

Construction of a Kleene Tree

$K_T = \{a \in 2^* \mid a \text{ does not clash with } \langle m_-, d_- \rangle\}$

K_T is a Kleene tree:

1. Clearly, decidable, inhabited, prefix-closed.
2. Unbounded: define $[a_0, \ldots, a_n]$ by

   $$a_j = \begin{cases} d_k & \text{if } j = m_k \text{ for some } k \leq n, \\ 0 & \text{otherwise.} \end{cases}$$

   Then $[a_0, \ldots, a_n]$ does not clash with $\langle m_-, d_- \rangle$.
3. Every path $\alpha \in 2^{\mathbb{N}}$ exits T_K: α and s are inconsistent, hence prefixes of α clash with $\langle m_-, d_- \rangle$ eventually.

Note: there is an enumeration $\ell : \mathbb{N} \to 2^*$ without repetitions of the leaves of T_K.

Cantor space and Baire space

The Cantor space $2^{\mathbb{N}}$ and Baire space $\mathbb{N}^{\mathbb{N}}$ are complete separable metric spaces, with metric (for both spaces)

$$d(\alpha, \beta) = 2^{-\min\{k \in \mathbb{N} \,\mid\, \alpha_k \neq \beta_k\}} .$$

Theorem

$2^{\mathbb{N}}$ *and* $\mathbb{N}^{\mathbb{N}}$ *are homeomorphic as metric spaces.*

Proof.

The homeomorphism $h : \mathbb{N}^{\mathbb{N}} \to 2^{\mathbb{N}}$ is defined by

$$h(\alpha) = \ell(\alpha_0)\ell(\alpha_1)\ell(\alpha_2)\cdots$$

□

Computing $2^{2^{\mathbb{N}}}$

$2^{2^{\mathbb{N}}}$ is the set of decidable subsets of decidable subsets.

$$2^{2^{\mathbb{N}}} = 2^{\mathbb{N}^{\mathbb{N}}} = 2^{\mathbb{N} \times \mathbb{N}^{\mathbb{N}}} = (2^{\mathbb{N}})^{\mathbb{N}^{\mathbb{N}}} = (\mathbb{N}^{\mathbb{N}})^{\mathbb{N}^{\mathbb{N}}} = \mathbb{N}^{\mathbb{N} \times \mathbb{N}^{\mathbb{N}}} = \mathbb{N}^{\mathbb{N}^{\mathbb{N}}} .$$

Remark: in sane models of computability, such as **Equ**, we have $2^{\mathbb{N}} \not\cong \mathbb{N}^{\mathbb{N}}$ and $2^{2^{\mathbb{N}}} = \mathbb{N}$.

Local non-compactness of $\mathbb{R}$

- The "middle-thirds" embedding $i : 2^{\mathbb{N}} \to [0,1]$, $i(\alpha) = \sum_{k=0}^{\infty} \frac{2\alpha_k}{3^{k+1}}$.
- The image $C = \mathrm{im}(i)$ is a closed located subset of $[0,1]$.
- The map $i \circ h : \mathbb{N}^{\mathbb{N}} \to [0,1]$ embeds $\mathbb{N}^{\mathbb{N}}$ as a closed located subset $C \subseteq [0,1]$.

Theorem (Specker sequence)

There exists a sequence $(a_n)_{n\in\mathbb{N}}$ *in* $[0,1]$ *without accumulation point.*

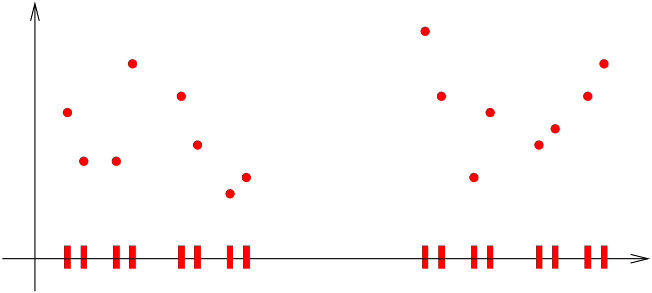
Proof.

The sequence $b_n = \lambda k.\, n$, is without accumulation point in $\mathbb{N}^{\mathbb{N}}$. Define $a_n = i(h(b_n))$. Then a_n is without accumulation point in C. Because C is closed and located, a_n is without accumulation point in $[0,1]$. □

Extending a continuous map $C \to \mathbb{R}$

Theorem

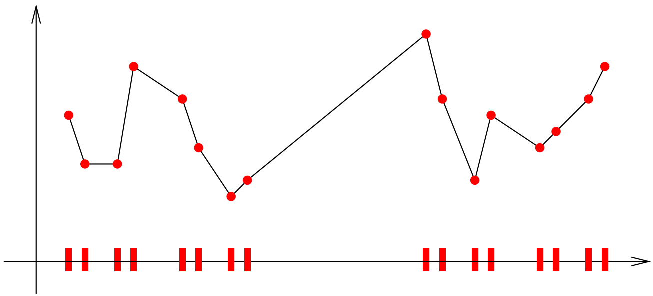
Every continuous $g : C \to \mathbb{R}$ can be extended to a continuous $\overline{g} : [0,1] \to \mathbb{R}$.



Extending a continuous map $C \to \mathbb{R}$

Theorem

Every continuous $g : C \to \mathbb{R}$ *can be extended to a continuous* $\overline{g} : [0, 1] \to \mathbb{R}$.



Unbounded continuous $f : [0,1] \to \mathbb{R}$

Theorem

There exists an unbounded continuous map $[0,1] \to \mathbb{R}$.

Proof.

- The map $g : \mathbb{N}^{\mathbb{N}} \to \mathbb{R}$, $g : \alpha \mapsto \alpha_0$ is unbounded and continuous.
- The map $g \circ h^{-1} \circ i^{-1} : C \to \mathbb{R}$ is unbounded and continuous.
- Extend $g \circ h^{-1} \circ i^{-1}$ to a continuous $f : [0,1] \to \mathbb{R}$. It is still unbounded.

$\square$

Conclusion

- The theme: as logicians, we should look for *elegant* presentations of theories we study. They can lead to new intuitions (and destroy old ones).
- These slides, and more, at `math.andrej.com`.
- We want food.